Quasi-static analysis

Consider a boom with n segments. The coordinate of the left end is represented by (x_L, y_L) and that of the the right end by (x_R, y_R). $\theta_i, i = 1, 2, \ldots, n$ represent the orientations of the segments.

Problem Statement. For a given feasible configuration described by $x_L, y_L, x_R, y_R, \theta_1, \theta_2, \cdots, \theta_n$, we are interested in finding a relationship between $\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}$ and $\dot{\theta_i}, i = 1, 2, \ldots, n$. This will enable us to develop a control law.

Analysis of degrees of freedom and forces using Newton's Equations

By hypothesis the configuration described by $x_L, y_L, x_R, y_R, \theta_1, \theta_2, \cdots, \theta_n$ is given and feasible, that is, it statisfies the shape constraints,

$$\begin{aligned} x_R &= x_L + \sum_{i=1}^{n} L_i \cos(\theta_i) \\ y_R &= y_L + \sum_{i=1}^{n} L_i \sin(\theta_i) \end{aligned} \tag{1}$$

By differentiating (1) once w.r.t. time, we get two equations liner in the dot terms that we can represent as follows,

$$\begin{aligned} C_1(\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}, \dot{\theta_1}, \dot{\theta_2}, \cdots, \dot{\theta_n}) &= 0 \\ C_2(\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}, \dot{\theta_1}, \dot{\theta_2}, \cdots, \dot{\theta_n}) &= 0 \end{aligned} \tag{2}$$

The unknown forces: There are the $2(n-1)$ forces acting at the $(n-1)$ joints between the n segments, and 2 forces on each end of the boom. So the total number of unknown forces is $2n + 2$.

Again, we have 3 force and moment balance equations for each segment (note each of these equations involve the unknown forces as well as the dot terms from the viscous forces).

Total number of equations: We get 2 equations (2) from the constraints and $3n$ force/moment equations. That is, a total of $3n + 2$ equations.
Total number of unknowns: $\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}, \dot{\theta_1}, \dot{\theta_2}, \cdots, \dot{\theta_n}$ consist of $n + 4$ unknowns, and $2n + 2$ unknown forces. That makes a total of $3n + 6$ unknowns. (Note: by hypothesis, shape variables $x_L, y_L, x_R, y_R, \theta_1, \theta_2, \cdots, \theta_n$ are given).

We can eliminate the $2n+2$ forces from the $3n+6$ equations. Thus we will be left with n equations and $n+4$ unknowns. That is, out of the $n+4$ unknowns, $\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}, \dot{\theta_1}, \dot{\theta_2}, \cdots, \dot{\theta_n}$, if we fix 4 of them, the rest will be uniquely determined by the n equations. For example, fixing $\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}$ (i.e. the velocities for the ends), the $\dot{\theta}_i$ will be uniquely determined (which is kind of intuitive for the quasistatic model).

However, our problem is to determine $\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}$ for a desired set of $\dot{\theta_i}$'s in order to develop a control law. Evidently, if $n=4$ this can be found uniquely, giving us arbirtary shape control. Otherwise we can use the following control laws:

Since the drag forces are linear in the dot terms, the n equations (after elemination of unknown forces) turn out to be linear in $\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}, \dot{\theta_1}, \dot{\theta_2}, \cdots, \dot{\theta_n}$ (note: linear, not affine). Thus these n equations can be written as,

$$A_{n\times 4}P + B_{n\times n}\dot{\theta} = 0 \tag{3}$$

where, $P = [\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}]^T$ and $\dot{\theta} = [\dot{\theta_1}, \dot{\theta_2}, \cdots, \dot{\theta_n}]^T$.
Note that the matrices $A_{n\times 4}$ and $B_{n\times n}$ contain only the shape variables $x_L, y_L, x_R, y_R, \theta_1, \theta_2, \cdots, \theta_n$, which are given.

The control law we use can now be written as,

$$P = -A^{+}_{n\times 4}B_{n\times n}K(\theta_D - \theta) \tag{4}$$

where, $(\cdot)^+$ is the MoorePenrose pseudoinverse, θ_D describes the desired shape to achive, and K is the gain matrix. The gain matrix can be dependent on the shape variable, but for now we are using $K = \epsilon I$.

A virtual work approach

In order to set up the equations relating $\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}$ and $\dot{\theta_i}$'s we can used a virtual work approach. Let $\theta_1, \theta_2, \cdots, \theta_n$ be the generalized coordinates. Thus we get n viruual forces, which need to be set to zero by our quasi-ststic assumption.

$$Q_i = \sum_k \mathbf{F_k}\frac{\partial \mathbf{r_k}}{\partial \theta_i}, \quad i = 1, \cdots, n \tag{5}$$

where the summations need to be done over all the external forces.

Note that each of these equations will involve the unknowns $\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}, \dot{\theta_1}, \dot{\theta_2}, \cdots, \dot{\theta_n}$ from the drag forces. They will also involve a total of two unknown forces from the applied forces at the ends of the boom. There are 4 forces being applied by the ships on the boom at its two ends. However one component of each of these forces do work and hence won't vanish as constraint force (imagine the two ends being on two rail tracks - the componet of forces perpendicular to the tracks at the point of contact do no work, but the ones along the tangents do work). Thus, *the above n equations involve $n+6$ unknowns - $\dot{x_L}, \dot{y_L}, \dot{x_R}, \dot{y_R}, \dot{\theta_1}, \dot{\theta_2}, \cdots, \dot{\theta_n}, F_1, F_2$.*

We also need to have the constrain equations (2), since those are not captured in the force equations.

Thus, we have $n + 2$ equations and $n + 6$ unknowns. This is consistent with the n equations and $n + 4$ unknown model we had in the previous section.