A General Continuous Inverse Kinematics Algorithm for a Planar Robot Arm

Subhrajit Bhattacharya*

Abstract

In this article we will describe a general algorithm (of computational complexity $O(n^2)$) for computing a continuous inverse kinematics for a planar robot arm with n segments, $\mathsf{IK} : \mathbb{R}_+ \to \mathfrak{R}, z \mapsto [\theta_{n-1}, \theta_{n-2}, \cdots, \theta_0]$. The algorithm, in essence, is a recursive one, that decomposes the planar arm into sub-arms, and incrementally construct the inverse kinematics of the full arm. The proposed algorithm can be used effectively for planning trajectories through critical points in the configuration space as is required in [BP14].

Full publication:

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Live demonstrations:

– http://subhrajit.net/wiki/index.php?xURL=arm_JS-HTML5
– http://hans.math.upenn.edu/~subhrabh/nowiki/robot_arm_JS-HTML5/arm.html

1 Introduction: Notations and Basic Observations

Configuration Space of a Robot Arm: Consider a n-segmented planar robot arm with segment lengths $r_0, r_1, \cdots, r_{n-1}$. The configuration space of this robot arm is $\mathbb{T}^n$, which we coordinatize using the angles that the segments of the arm make with the positive X axis (see Figure 1). Thus, $[\theta_{n-1}, \theta_{n-2}, \cdots, \theta_0]$, with $\theta_i \in \mathbb{S}^1$, gives an unique configuration of the arm.

Throughout the paper we will assume that the end effector of the arm can be at points $q \in (\mathbb{R}^2 - \{0\})$, where $\{0\}$ is the location of the fixed base of the arm. That is, we eliminate the cases when the end effector of the robot arm coincides with the base (equivalently, the base-length, z, vanishes). Thus the configuration space that is of interest to us is $(\mathbb{T}^n - \tilde{O})$, where $\mathbb{T}^n \supset \tilde{O} = \{[\theta_{n-1}, \theta_{n-2}, \cdots, \theta_0] \in \mathbb{T}^n \mid \sum_{i=0}^{n-1} r_i \sin(\theta_i) = \sum_{i=0}^{n-1} r_i \cos(\theta_i) = 0\}$. We call this the *non-singular configuration space*.

Restricted Configuration Space: It is sufficient to study the configuration space of the arm up to an equivalence of rotations of the entire arm about the base. In fact the equivalence can be extended to translations and scalings of the arm as discussed in [KM94, Jag92]. However, the base of the robot arm being fixed at a point, and the segment lengths being fixed for a given robot arm, those equivalences are not relevant in our problem. In particular, we define the *restricted configuration space* as $\mathfrak{R} = (\mathbb{T}^n - \tilde{O})/\sim$, where the equivalence '$\sim$' deems two arm configurations to be equivalent if one can be rotated about the base point to obtain the other. We represent an element in $\mathfrak{R}$ by the configuration in the equivalence class that

*Department of Mathematics, University of Pennsylvania, Philadelphia PA 19104. email: subhrabh@math.upenn.edu

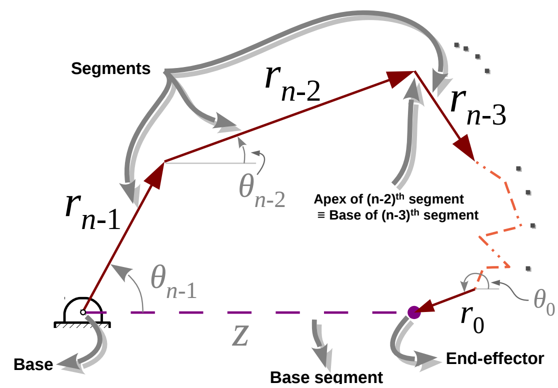


Figure 1: A general planar arm at a configuration that is in its restricted configuration space, $\mathfrak{R} \subset \mathbb{T}^n$.

has its end effector placed along the positive X axis (*i.e.*, $[\theta_{n-1}, \theta_{n-2}, \cdots, \theta_0]$ such that $\sum_{i=0}^{n-1} r_i \sin(\theta_i) = 0,\ \sum_{i=0}^{n-1} r_i \cos(\theta_i) \neq 0$ — Figure 1), and thus focus our inverse kinematics design problem only on $\mathfrak{R}$. For computing to an inverse kinematics for the entire configuration space, given an end effector position, we can simply set our positive X axis along the line joining the base of the arm and the end effector, and thus use the designed IK for the restricted configuration space.

It is easy to observe that the entire non-singular configuration space is the trivial $\mathbb{S}^1$-bundle, $(\mathbb{T}^n - \tilde{O}) \approxeq \mathfrak{R} \times \mathbb{S}^1$. Likewise, the entire end-effector space is the trivial $\mathbb{S}^1$-bundle $(\mathbb{R}^2 - \{0\}) \approxeq \mathbb{R}_+ \times \mathbb{S}^1$. On the other hand, a continuous inverse kinematics designed on the restricted configuration space, $IK : \mathbb{R}_+ \to \mathfrak{R}$, is a map between base spaces of these trivial bundles. Thus, a pull-back by IK of the end-effector bundle will give the bundle map $\widehat{IK} : (\mathbb{R}^2 - \{0\}) \to (\mathbb{T}^n - \tilde{O})$, which is an inverse kinematics on the entire configuration space. This discussion can be summarized in the following diagram with $\widehat{IK} = IK \times \text{Id}_{\mathbb{S}^1}$:

$$\begin{array}{ccccc} (\mathbb{T}^n - \tilde{O}) & \approxeq & \mathfrak{R} & \times & \mathbb{S}^1 \\ \uparrow \widehat{IK} & & \uparrow IK & & \uparrow \text{Id}_{\mathbb{S}^1} \\ (\mathbb{R}^2 - \{0\}) & \approxeq & \mathbb{R}_+ & \times & \mathbb{S}^1 \end{array}$$

Equivalence of Configuration Spaces Under Segment Reordering: Suppose the lengths of the segments of a robot arm, R, in sequence, starting from the base, are $r_{n-1}, r_{n-2}, \cdots, r_0$. We call these the $(n-1)^{th}, (n-2)^{th}, (n-3)^{th}, \cdots$, and 0^{th} segments. Let the arm's restricted configuration space be $\mathfrak{R}$. A arm, R', with segment lengths $r_{\sigma(n-1)}, r_{\sigma(n-2)}, \cdots, r_{\sigma(0)}$ (respectively, starting from the base), where σ is a permutation of the ordered set $[n-1, n-2, \cdots, 0]$, will have a restricted configuration space, $\mathfrak{R}'$, that is diffeomorphic to $\mathfrak{R}$ (the diffeomorphism being given by the permutation map $\sigma : \mathfrak{R} \to \mathfrak{R}',\ [\theta_{n-1}, \theta_{n-2}, \cdots, \theta_0] \mapsto [\theta_{\sigma(n-1)}, \theta_{\sigma(n-2)}, \cdots, \theta_{\sigma(0)}]$).

This diffeomorphism preserves base length: $D_R([\theta_*]) = D_{R'} \circ \sigma([\theta_*])$ (where, $D_R : [\theta_{n-1}, \theta_{n-2}, \cdots, \theta_0] \mapsto \sum_{i=0}^{n-1} r_i \cos(\theta_i)$, is the base-length map on the restricted configuration space of robot arm R) — Figure 2. Furthermore, every continuous inverse kinematics for arm R, $IK : \mathbb{R}_+ \to \mathfrak{R},\ z \mapsto [\theta_{m-1}, \theta_{m-2}, \cdots, \theta_0]$, gives a continuous inverse kinematics for the arm R' via the permutation map, $IK' = \sigma \circ IK : \mathbb{R}_+ \to \mathfrak{R}'$. Thus, although $\mathfrak{R}$ and $\mathfrak{R}'$ are the restricted configuration spaces of two different arms, as far as forward and inverse kinematics are concerned, it does not matter which arm we use for computation — results from one can be used for the other, with the angles only permuted.

2 Inverse Kinematics Algorithm

Consider a robot arm, R, with n segments. In this section we will describe a general algorithm (of computational complexity $O(n^2)$) for computing a continuous inverse kinematics $\mathsf{IK} : \mathbb{R}_+ \to \mathfrak{R}, z \mapsto [\theta_{n-1}, \theta_{n-2}, \cdots, \theta_0]$. Consider the arm configuration described by the orientations, $\theta_p \in \mathbb{S}^1,\ p = 0, 1, \cdots, n-1$, that the p^{th} segment makes with the positive X axis as described in Figure 3(a). Let x_p be the distance

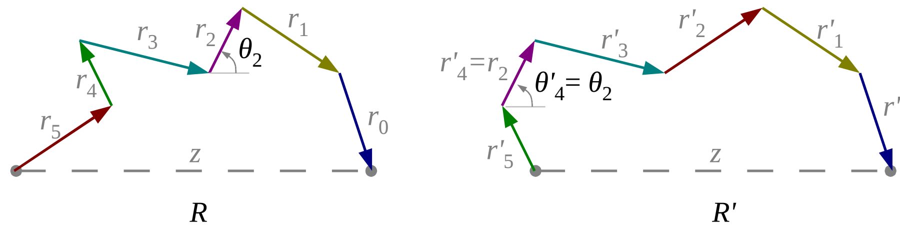
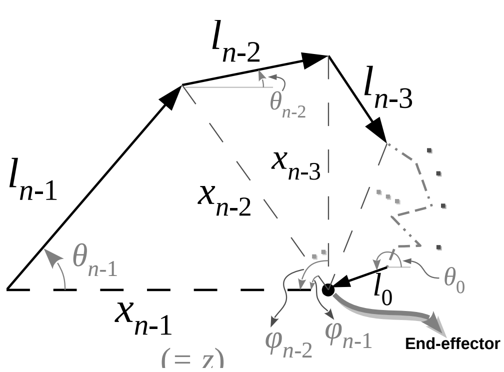


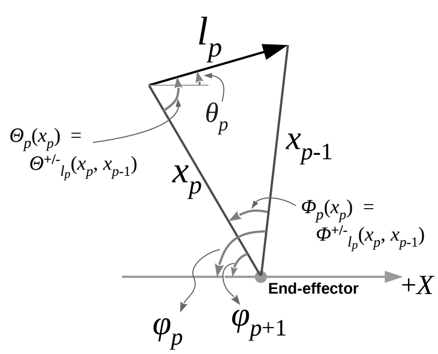
Figure 2: Two arms R and R', with the segments permuted. Any computations on one of the arms (say, angles computed using an inverse kinematics) can be used, via a permutation, for the other arm.

of the end-effector from the base of the p^{th} segment, as shown. Since $z = x_{n-1}$, an inverse kinematics can be described as a set of maps $x_p \mapsto x_{p-1},\ p = n-1, n-2, \cdots, 1$, and the corresponding description of the configurations of the triangles formed by the sides of lengths l_p, x_p and x_{p-1} as ilustrated in Figure 3(b).

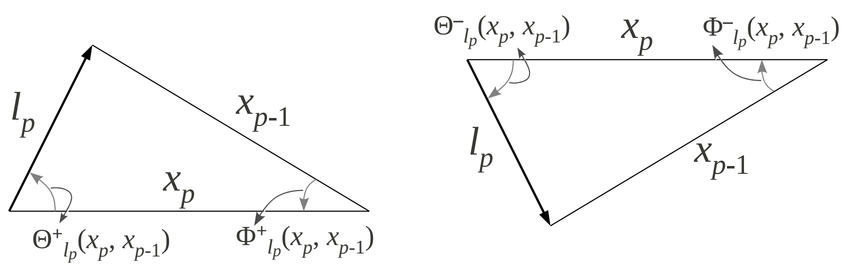
As a first step towards designing the inverse kinematics, we relate these lengths with the angles subtended by the p^{th} segment at the end-effector. Figure 3(c) shows the two possible triangles subtended by the p^{th} segment at the end effector such that the two other sides of the triangle are x_p and x_{p-1} (the order being consistent with the direction in which the segment points).



(a) A general robot arm configuration.



(b) The triangle subtended by the p^{th} segment at the end-effector.



(c) The functions $\Theta^{+/-}_{l_p}, \Phi^{+/-}_{l_p} : \mathbb{R}^2_+ \to \mathbb{S}^1$ respectively give the angle at the base of the p^{th} segment and that subtended at the end-effector by it as a function of the other side lengths, x_p and x_{p-1}. The '$+/-$' indicate two possible configurations.

Figure 3: .

$$\Theta^{\pm}_{l_p}(x_p, x_{p-1}) \;=\; \pm \arccos\left(\frac{l_p^2 + x_p^2 - x_{p-1}^2}{2 l_p x_p}\right) \;,\quad \Phi^{\pm}_{l_p}(x_p, x_{p-1}) \;=\; \pm \arccos\left(\frac{x_p^2 + x_{p-1}^2 - l_p^2}{2 x_p x_{p-1}}\right) \tag{1}$$

The range of $\Theta^+_{l_p}$ and $\Phi^+_{l_p}$ is $[0,\pi] \subset \mathbb{S}^1$, while that of $\Theta^-_{l_p}$ and $\Phi^-_{l_p}$ is $\{\pi\} \cup (-\pi, 0] \subset \mathbb{S}^1$. These functions are continuous, except when any side of the triangle is of length zero. By hypothesis $l_p > 0$. Hence, in order to ensure that we don't encounter singularities or discontinuities in designing the inverse kinematics, we will steer clear of $x_p = 0$ or $x_{p-1} = 0$.

2.1 The Relationship Between Ranges of Partial Arms

We determine the range of values that x_p can assume using a recursive argument by relating the range of x_p with that of x_{p-1}. Clearly $x_0 = l_0$ is always constant. Suppose x_{p-1} can assume all values in the interval $[\underline{x}_{p-1}, \overline{x}_{p-1}]$. Using triangle inequality for the triangle subtended by the p^{th} segment,

$$|x_{p-1} - l_p| \quad \leq \quad x_p \quad \leq \quad x_{p-1} + l_p \tag{2}$$

it is easy to check that the range of values that x_p can assume, $[\underline{x}_p, \overline{x}_p]$, is given by,

$$\begin{aligned} \underline{x}_p \;&=:\; \underline{R}(\underline{x}_{p-1}, \overline{x}_{p-1}; l_p) \;=\; \begin{cases} \underline{x}_{p-1} - l_p, & \text{if } \; l_p \leq \underline{x}_{p-1} \\ 0, & \text{if } \; \underline{x}_{p-1} < l_p \leq \overline{x}_{p-1} \\ l_p - \overline{x}_{p-1}, & \text{if } \; \overline{x}_{p-1} < l_p \end{cases} \\ \overline{x}_p \;&=:\; \overline{R}(\underline{x}_{p-1}, \overline{x}_{p-1}; l_p) \;=\; l_p \;+\; \overline{x}_{p-1} \end{aligned} \tag{3}$$

Thus, using $\underline{x}_0 = \overline{x}_0 = l_0$ and the recursive relation of (3), we can work out $\underline{x}_p$ and $\overline{x}_p$ for all $p = 0, 1, 2, \cdots, n-1$ (the computational complexity being linear in n). Moreover, since $\overline{x}_p$ or $\underline{x}_p$ depend only on the values in the set $\{l_0, l_1, \cdots, l_p\}$, we re-write the above recursive relations using the following simplified notations:

$$\begin{aligned} \underline{x}_p &= \underline{R}(\{l_0, l_1, \cdots, l_p\}) \\ \overline{x}_p &= \overline{R}(\{l_0, l_1, \cdots, l_p\}) \end{aligned} \tag{4}$$

Proposition 1 (The closed-form expressions for $\underline{R}$ and $\overline{R}$)**.**

1. *$\overline{R}(\{l_0, l_1, \cdots, l_p\}) = \sum_{j=0}^{p} l_j$*
2. *Since, due to the discussion of Section 1, the order of the elements in the set $\{l_0, l_1, \cdots, l_p\}$ does not change the value of $\underline{R}(\{l_0, l_1, \cdots, l_p\})$, without loss of generality we assume $l_p \geq l_{p-1} \geq \cdots \geq l_1 \geq l_0$. Then, $\underline{R}(\{l_0, l_1, \cdots, l_p\}) = \begin{cases} l_p - \sum_{j=0}^{p-1} l_j, & \text{if } l_p > \sum_{j=0}^{p-1} l_j, \\ 0, & \text{otherwise.} \end{cases}$*

Proof. '1.' follows trivially from the recursive expression for $\overline{x}_p$ in (3).

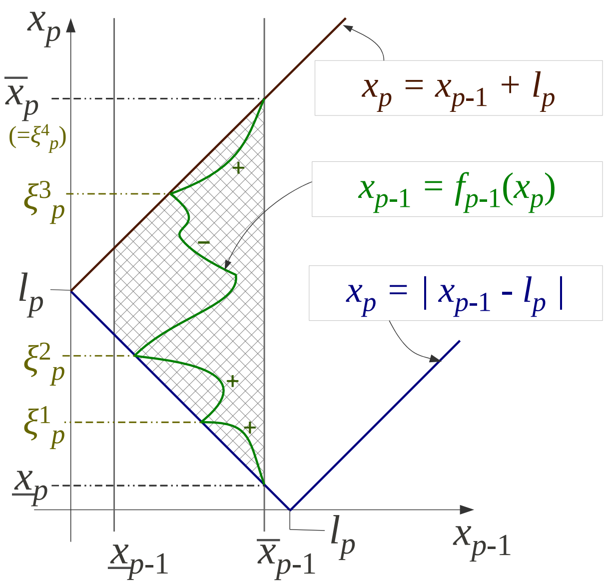
We prove '2.' by considering the value of $\underline{R}(\{l_0, l_1, \cdots, l_p\})$ separately for the two cases.
Case $l_p > \sum_{j=0}^{p-1} l_j$: Using (3),

$$\begin{aligned} &\underline{R}(\{l_0, l_1, \cdots, l_p\}) \\ &= l_p - \overline{x}_{p-1} \qquad [\text{where, } \overline{x}_{p-1} = \overline{R}(\{l_0, l_1, \cdots, l_{p-1}\}), \text{ and since } \overline{R}(\{l_0, l_1, \cdots, l_{p-1}\}) = \textstyle\sum_{j=0}^{p-1} l_j < l_p] \\ &= l_p - \textstyle\sum_{j=0}^{p-1} l_j \end{aligned}$$

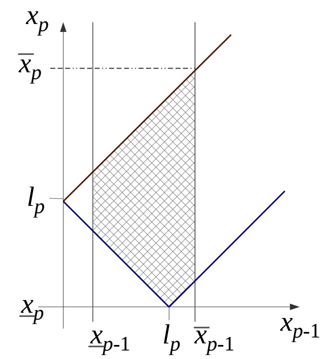
Case $l_p \leq \sum_{j=0}^{p-1} l_j$: This condition implies $l_p \leq \overline{R}(\{l_0, l_1, \cdots, l_p\}) =: \overline{x}_{p-1}$. Furthermore, $\underline{x}_{p-1}$ is the minimum value of the base of an arm with segment lengths $\{l_0, l_1, \cdots, l_{p-1}\}$. It is easy to check that since the values are ordered, we have $0 \leq l_{p-1} - l_{p-2} + l_{p-3} - l_{p-4} + \cdots l_0 \leq l_{p-1}$, and the base-length of $(l_{p-1} - l_{p-2} + l_{p-3} - \cdots)$ is achievable by the arm with segment lengths $\{l_0, l_1, \cdots, l_{p-1}\}$ (in the configuration when all the segments are aligned along a single line, but with alternating orientations). Thus for its minimum possible value we must have $\underline{x}_{p-1} \leq (l_{p-1} - l_{p-2} + l_{p-3} - \cdots) \leq l_{p-1}$. Thus we have $\underline{x}_{p-1} \leq l_{p-1} \leq \overline{x}_{p-1}$. Using (3) we thus immediately have

$$\underline{R}(\{l_0, l_1, \cdots, l_p\}) \quad = \quad 0$$

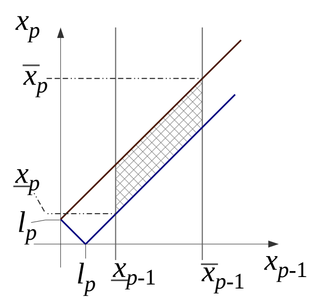
□



(a) $\mathcal{F}_{p-1}$ of type $l_p \geq \overline{x}_{p-1}$. An inverse kinematics component function, f_{p-1}, and a *sign switching set*, $S_p = \{\xi_p^1, \xi_p^2, \xi_p^3, \xi_p^4\}$, are also shown. The '+' and '−' indicate a choice of *sign assignment*.



(b) $\mathcal{F}_{p-1}$ of type $\overline{x}_{p-1} \geq l_p \geq \underline{x}_{p-1}$.



(c) $\mathcal{F}_{p-1}$ of type $\underline{x}_{p-1} \geq l_p$.

Figure 4: Relation between $[\underline{x}_p, \overline{x}_p]$ and $[\underline{x}_{p-1}, \overline{x}_{p-1}]$, and the feasibility set $\mathcal{F}_{p-1}$ (hatched region). An example of a IKCF-IKCSSS-IKCSA tuple is also shown in (a).

Corollary 1.

$$\underline{R}(\{l_0, l_1, \cdots, l_p\}) \quad \leq \quad | \pm l_0 \pm l_1 \pm \cdots \pm l_p|$$

for any assignment of '+' or '−' for the '±' signs on the right hand side of the inequality.

> *Proof.* Once again we assume, without the loss of generality, that $l_p \geq l_{p-1} \geq \cdots \geq l_1 \geq l_0$. The R.H.S. is always non-negative. The L.H.S. is either 0 or $l_p - \sum_{j=0}^{p-1} l_j$ (when $l_p > \sum_{j=0}^{p-1} l_j$). If L.H.S. is 0 there is nothing to prove. However, if $l_p > \sum_{j=0}^{p-1} l_j$, we note that the R.H.S. can be re-written as $l_p \pm l_{p-1} \pm l_{p-2} \pm \cdots \pm l_0$, for any assignment of '+' or '−' for the '±'s. This clearly is greater than or equal to $l_p - l_{p-1} - l_{p-2} - \cdots - l_0 \geq \underline{R}(\{l_0, l_1, \cdots, l_p\})$. □

For convenience we define the following:

Definition 1 (Feasible set for the graph of a function relating x_p and x_{p-1}.)**.** We define

$$\mathcal{F}_{p-1} \quad = \quad \left\{ (x_{p-1}, x_p) \;\middle|\; x_p \in [\underline{x}_p, \overline{x}_p], \quad x_{p-1} \in [\underline{x}_{p-1}, \overline{x}_{p-1}] \;\text{ and }\; |x_{p-1} - l_p| \leq x_p \leq x_{p-1} + l_p \right\}$$

This is the feasible set, due to inequality (2), inside which the graph of any function relating x_p and x_{p-1} should lie.

We can thus classify the possible "shapes" of $\mathcal{F}_{p-1}$ into three types (not mutually disjoint): *i.* type $l_p \geq \overline{x}_{p-1}$, *ii.* type $\overline{x}_{p-1} \geq l_p \geq \underline{x}_{p-1}$ and *iii.* type $\underline{x}_{p-1} \geq l_p$ (see Figure 4).

2.2 Inverse Kinematics Components

Definition 2 (IKCF)**.** We define a $(p-1)^{th}$ "inverse kinematics component function" (IKCF) as a continuous function $f_{p-1} : [\underline{x}_p, \overline{x}_p] \rightarrow [\underline{x}_{p-1}, \overline{x}_{p-1}]$ such that $(f_{p-1}(x_p), x_p) \in \mathcal{F}_{p-1}$ and $f_{p-1}(x_p) > 0, \; \forall x_p \in [\underline{x}_p, \overline{x}_p]$.

It is always possible to construct such a IKCF if $l_p, \overline{x}_p > 0$, since then $\mathcal{F}_{p-1}$ is a non-empty convex polygon (Figure 4). Clearly, $f_0 : [\underline{x}_1, \overline{x}_1] \to \{l_0\}$ is a constant function (since $\underline{x}_0 = \overline{x}_0 = l_0$).

Definition 3 (IKCSSS)**.** For a given IKCF, $f_{p-1} : [\underline{x}_p, \overline{x}_p] \to [\underline{x}_{p-1}, \overline{x}_{p-1}]$, let

$$\Xi(f_{p-1}) \;=\; \left\{ x_p \in [\underline{x}_p, \overline{x}_p] \;\middle|\; \text{Either } x_p = |f_{p-1}(x_p) - l_p| \text{ or } x_p = f_{p-1}(x_p) + l_p. \right\}$$

Note that this set contains at least one point, namely, $\overline{x}_p$. A finite countable subset of points, $S_p = \{\xi_p^1, \xi_p^2, \cdots, \xi_p^{h_p-1}, \xi_p^{h_p} = \overline{x}_p\} \subseteq \Xi(f_{p-1})$, such that $\overline{x}_p$ is an element of the set, is called an "inverse kinematics component sign switching set" (IKCSSS). Without loss of generality we assume S_p to be an ordered set with $\xi_p^1 \leq \xi_p^2 \leq \cdots \leq \xi_p^{h_p-1} \leq \xi_p^{h_p} = \overline{x}_p$ (see Figure 4(a)). The points in such a set have the property that $\Theta_{l_p}^{+}(\xi, f_{p-1}(\xi)) = \Theta_{l_p}^{-}(\xi, f_{p-1}(\xi))$ and $\Phi_{l_p}^{+}(\xi, f_{p-1}(\xi)) = \Phi_{l_p}^{-}(\xi, f_{p-1}(\xi)),\ \forall \xi \in S_p$ (Figure 3(c)).

Definition 4 (IKCSA)**.** Given a IKCF, f_{p-1}, and a corresponding valid choice of IKCSSS, S_p, we define a "inverse kinematics component sign assignment" (IKCSA) as a map $\mathbf{sg}_p : S_p \to \{\text{`+'}, \text{`−'}\}$.

Definition 5 (IKCISF)**.** The reason of defining the IKCSSS and a corresponding IKCSA is that now we can assign a 'sign' to the intervals $[\underline{x}_p, \xi_p^1)$, $[\xi_p^1, \xi_p^2)$, $[\xi_p^2, \xi_p^3)$, $\cdots$, $[\xi_p^{h_p-1}, \overline{x}_p]$, as follows:

$$\begin{aligned} \mathscr{S}_{S_p, \mathbf{sg}_p} \;&:\; [\underline{x}_p, \overline{x}_p] \to \{\text{`+'}, \text{`−'}\} \\ &:\; x_p \mapsto \begin{cases} \mathbf{sg}_p(\xi_p^1), & \text{if } \underline{x}_p \leq x_p < \xi_p^1 \\ \mathbf{sg}_p(\xi_p^2), & \text{if } \xi_p^1 \leq x_p < \xi_p^2 \\ \vdots & \\ \mathbf{sg}_p(\xi_p^{h_p}), & \text{if } \xi_p^{h_p-1} \leq x_p \leq \xi_p^{h_p} = \overline{x}_p \end{cases} \end{aligned} \tag{5}$$

This function, $\mathscr{S}_{S_p, \mathbf{sg}_p}$, defined using a given pair of IKCSSS and IKCSA, will be refereed to as "inverse kinematics component interval sign function" (IKCISF), or simply the "interval sign function". This is illustrated using the '+' or '−' in Figure 4(a).

[*Note:* If $\xi_p^{j-1} = \xi_p^j$, then $\mathbf{sg}_p(\xi_p^j)$ is essentially not used in the construction of $\mathscr{S}_{S_p, \mathbf{sg}_p}$.]

We will write $\{f_{p-1}, \mathscr{S}_{S_p, \mathbf{sg}_p}\}$ to indicate a choice of IKCF, and an interval sign assignment due to the choice of a corresponding IKCSSS-IKCSA pair. Due to the following lemma, a choice of these determines a continuous map from the range of x_p to the space of configuration for the triangle with sides l_p, x_p and $f_{p-1}(x_p)$ (Figure 3(c)).

Lemma 1. *Given a IKCF-IKCSSS-IKCSA tuple, $\{f_{p-1}, \mathscr{S}_{S_p, \mathbf{sg}_p}\}$, the following functions, $\Theta_p, \Phi_p : [\underline{x}_p, \overline{x}_p] - \{0\} \to \mathbb{S}^1$, are continuous:*

$$\begin{aligned} \Theta_p(x_p) \;&:=\; \Theta_{l_p}^{\mathscr{S}_{S_p, \mathbf{sg}_p}(x_p)}(x_p, f_{p-1}(x_p)) \\ \Phi_p(x_p) \;&:=\; \Phi_{l_p}^{\mathscr{S}_{S_p, \mathbf{sg}_p}(x_p)}(x_p, f_{p-1}(x_p)) \end{aligned} \tag{6}$$

where, by $\Theta_{l_p}^s$ we mean $\Theta_{l_p}^{+}$ or $\Theta_{l_p}^{-}$, depending on whether s is '+' or '−' (and likewise for Φ).

Proof. As described earlier, $\Theta_{l_p}^{+/-}, \Phi_{l_p}^{+/-} : \mathbb{R}_+^2 \to \mathbb{S}^1$ are continuous functions in their respective domains except where either of their inputs are zero. Again, by the construction of IKCF, f_{p-1} is a continuous function which does not have zero in its image. Thus for all $x_p \in [\underline{x}_p, \overline{x}_p] - \{0\}$, $\Theta_{l_p}^{+/-}(x_p, f_{p-1}(x_p))$ and $\Phi_{l_p}^{+/-}(x_p, f_{p-1}(x_p))$ are continuous functions.

Next recall that $\Theta_{l_p}^{+}(\xi_p^i, f_{p-1}(\xi_p^i)) = \Theta_{l_p}^{-}(\xi_p^i, f_{p-1}(\xi_p^i))$ and $\Phi_{l_p}^{+}(\xi_p^i, f_{p-1}(\xi_p^i)) = \Phi_{l_p}^{-}(\xi_p, f_{p-1}(\xi_p^i))$ for all $\xi_p^i \in S_p$. Thus $\Theta_p, \Phi_p : \mathbb{R}_+ \to \mathbb{S}^1$ are made up of piece-wise continuous functions that agree at the points where the constituent functions are pieced together. This concludes the proof. □

In the above Lemma, Φ_p and Θ_p are defined for $p = 1, 2, \cdots, m-1$. We extend the definition for $p = 0$ by letting $\Phi_0, \Theta_0 : \{l_0\} \to 0 \in \mathbb{S}^1$.

2.3 The Inverse Kinematics Algorithm

Thus, given IKCF-IKCSSS-IKCSA tuples, $\{f_{p-1}, \mathscr{S}_{S_p,\mathbf{sg}_p}\}$, for $p = 1, 2, \cdots, n-1$, and given a base-length, $z \equiv x_{n-1} > 0$, we construct a continuous inverse kinematics for the entire arm as follows:

For a given value of $z \equiv x_{n-1}$ (the base-length), we can compute

$$x_k \;=\; F_k(z) \;:=\; f_k \circ f_{k+1} \circ f_{k+2} \circ \cdots \circ f_{n-2}(z) \tag{7}$$

Clearly, F_k are continuous $\forall k = 0, 1, 2, \cdots, n-2$. Moreover, by the definition of IKCF, $z > 0 \Rightarrow x_k > 0$.

Thus, a complete configuration for the arm is determined by the following expression for the orientation of the p^{th} segment,

$$\begin{aligned} \theta_p \quad &= \quad \Theta_p(x_p) \;-\; \phi_{p+1} \quad = \quad \Theta_p(x_p) \;-\; \sum_{k=p+1}^{n-1} \Phi_k(x_k) \qquad \text{(refer to Figure 3(b))} \\ &= \quad \Theta_q^{\mathscr{S}_{S_q,\mathbf{sg}_q}(x_q)}(x_q, f_{q-1}(x_q)) \;-\; \sum_{k=q+1}^{n-1} \Phi_k^{\mathscr{S}_{S_k,\mathbf{sg}_k}(x_k)}(x_k, f_{k-1}(x_k)) \end{aligned} \tag{8}$$

Since $x_k > 0$, using Lemma 1 it follows that θ_q varies continuously with the base-length, z. Furthermore, the fact that $[\theta_{n-1}, \theta_{n-2}, \cdots, \theta_0] \in \mathfrak{R} \subset \mathbb{T}^n$ follows from our very construction (see Figure 3). This can also be proved explicitly by simplifying the trigonometric expressions $x_e = \sum_{j=0}^{n-1} l_j \cos(\theta_j)$, $y_e = \sum_{j=0}^{n-1} l_j \sin(\theta_j)$ using the trigonometric formulae for $\Theta^{+/-}$ and $\Phi^{+/-}$ to show that $y_e = 0$. This we omit, considering it a simple but lengthy exercise.

We thus have the following proposition:

Proposition 2. *The map from base-length, $z > 0$, to the space of arm configurations described by the segment orientations as determined by (8) (along with (7)) is a continuous map from $\mathbb{R}_+$ to the restricted configuration space, $\mathfrak{R}$. Since the map is completely determined by the tuple $\{f_{p-1}, \mathscr{S}_{S_p,\mathbf{sg}_p}\}$, $p = 1, 2, \cdots, n-1$, for brevity we write this map as*

$$\begin{aligned} \mathsf{IK}_{\{f_{*-1}, \mathscr{S}_{S_*,\mathbf{sg}_*}\}} \;:\; & \mathbb{R}_+ \to \mathfrak{R} \\ & z \mapsto [\theta_{n-1}, \theta_{n-2}, \cdots, \theta_0] \end{aligned}$$

3 Some Particular IKCFs

In this section we will describe three types of functions, , that will be particularly useful for constructing the desired IKCFs, $f_{p-1} : [\underline{x}_p, \overline{x}_p] \to [\underline{x}_{p-1}, \overline{x}_{p-1}]$, in the next sections:

$$\begin{aligned} \mathsf{MIN}_{\{\underline{x}_{p-1}, l_p\}} \;&:\; [\underline{x}_p, \overline{x}_p] \to [\underline{x}_{p-1}, \overline{x}_{p-1}], \\ &:\; x_p \;\mapsto\; \max(\underline{x}_{p-1},\; |x_{p-1} - l_p|) \end{aligned} \tag{9}$$

$$\begin{aligned} \mathsf{MAX}_{\{\overline{x}_{p-1}, l_p\}} \;&:\; [\underline{x}_p, \overline{x}_p] \to [\underline{x}_{p-1}, \overline{x}_{p-1}], \\ &:\; x_p \;\mapsto\; \min(\overline{x}_{p-1},\; x_{p-1} + l_p) \end{aligned} \tag{10}$$

It is important that we do not use $\mathsf{MIN}_{\{\underline{x}_{p-1}, l_p\}}$ if $\underline{x}_{p-1} = 0$, since that will violate the positivity condition for the IKCF.

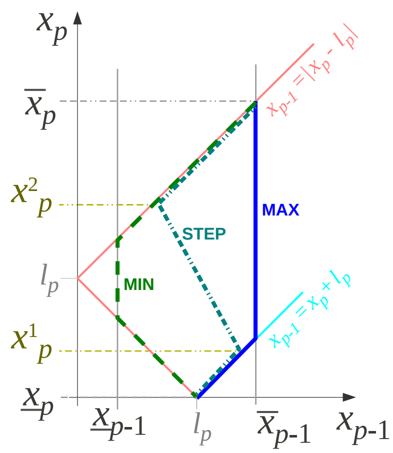


Figure 5: The functions MIN, STEP and MAX that we use to construct IKCFs.

Also, for given $x_p^1, x_p^2 \in [\underline{x}_p, \overline{x}_p]$, with $x_p^1 < x_p^2$, we define

$$\begin{aligned} \mathsf{STEP}_{\{\underline{x}_{p-1},\overline{x}_{p-1},l_p,x_p^1,x_p^2\}} \; &: \; [\underline{x}_p, \overline{x}_p] \to [\underline{x}_{p-1}, \overline{x}_{p-1}], \\ &: \; x_p \; \mapsto \; \begin{cases} \mathsf{MAX}_{\{\overline{x}_{p-1},l_p\}}(x_p), & \text{if } x_p \leq x_p^1 \\ \frac{x_p^2 - x_p}{x_p^2 - x_p^1} \mathsf{MAX}_{\{\overline{x}_{p-1},l_p\}}(x_p) + \frac{x_p - x_p^1}{x_p^2 - x_p^1} \mathsf{MIN}_{\{\underline{x}_{p-1},l_p\}}(x_p), & \text{if } x_p^1 < x_p < x_p^2 \\ \mathsf{MIN}_{\{\underline{x}_{p-1},l_p\}}(x_p), & \text{if } x_p^2 \leq x_p \end{cases} \end{aligned} \tag{11}$$

Once again we will be careful not to use $\mathsf{STEP}_{\{\underline{x}_{p-1},\overline{x}_{p-1},l_p,x_p^1,x_p^2\}}$ when $x_p^2 = l_p$ and $\underline{x}_{p-1} = 0$. It is however worth noting that even when x_p^1 or x_p^2 are not in $[\underline{x}_p, \overline{x}_p]$, this still defines a valid IKCF.

Referring to Figure **??**, the function MIN essentially returns the minimum of the possible values of x_{p-1} (within the feasible region – the *hatched* region of Figure 4) for a given x_p, while MAX returns the maximum. STEP on the other hand, returns the minimum when x_p is greater than x_p^2, and maximum when x_p is less than x_p^1, while linearly interpolating in between.

References

[BP14] Subhrajit Bhattacharya and Mihail Pivtoraiko. A classification of configuration spaces of planar robot arms for a continuous inverse kinematics problem. *Acta Applicandae Mathematicae*, online first, September 2014. DOI: s10440-014-9973-1.

[Jag92] Beat Jaggi. *Configuration spaces of point sets with distance constrains.* PhD thesis, University of Bern, 1992. (See the summarized results that appear in the note *"Configuration Spaces of Planar Polygons"* by the same author).

[KM94] Michael Kapovich and John Millson. On the moduli space of polygons in the euclidean plane. *Journal of Differential Geometry*, 42:133–164, 1994.