Configuration Spaces and Topology †

Supplementary Notes – Part 1

Introduction to Configuration Spaces and Point-set Topology

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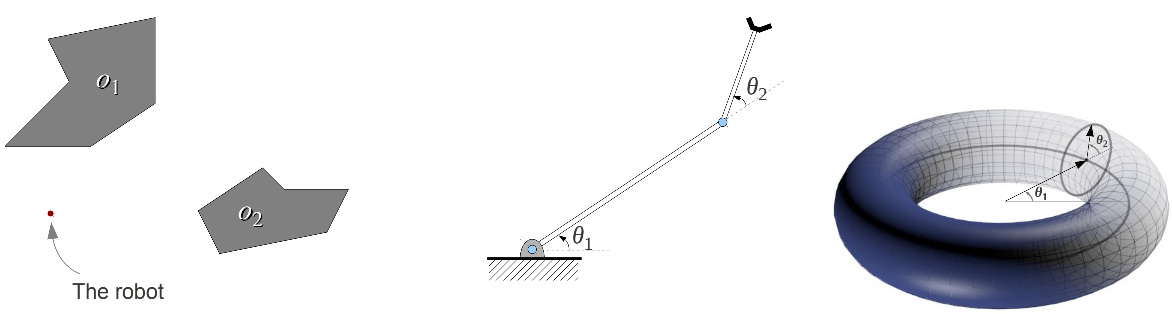
1 Introduction: Configuration Spaces

Many problems in robotics involve a configuration space. Configuration space or *C-space* of a robot or a system of robots is the abstract space of possible states or configurations that the system can attain. Thus each point in the C-space corresponds to a possible state of the robot(s) in the real environment. Robotics problems, especially planning problems, typically involve navigation of the system through the C-space space in order to achieve certain tasks or objectives. This translates to finding a 1-dimensional curve (a trajectory) in the C-space that the system needs to follow. Typically C-spaces are smooth manifolds and any curve on it is a possible trajectory for the system. However, presence of kinematic and dynamic constraints may require that the tangent at any point on the trajectory lies within a specific subset of the tangent space at the point of the C-space. Furthermore, in presence of a metric in the C-space or a more general measure for 1-dimensional curves in the space, one can talk about optimality of the trajectory.

Typically the C-space of a system can be parametrized by the different state variables corresponding to the different degrees of freedom. For example, the configuration space of a single point mobile robot navigating in a unbounded 2-dimensional flat plane with obstacles is simply $\mathbb{R}^2 - \mathcal{O}$, where $\mathcal{O}$ represents the set of points on the plane that make up the obstacles (Figure 1(a)). Similarly, the configuration space of a planar robot arm with two links and no joint angle limits (Figure 1(b)) is a *torus*, $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, each point on which correspond to an unique pair of joint angles θ_1, θ_2 (Figure 1(c)).



(a) The configuration space of a point robot navigating on a plane with obstacles o_1 and o_2 is $\mathbb{R}^2 - (o_1 \cup o_2)$

(b) A 2-link robotic arm is described by the state variables θ_1 and θ_2.

(c) The configuration space of a 2-link robotic arm is the 2-torus.

Figure 1: Examples of simple configuration spaces.

One can generalize the notion of C-space for a system with multiple robots. For example, if $\mathcal{C} = \mathbb{R}^2 - \mathcal{O}$ is the configuration space of a point robot as described in Figure 1(a), the presence of n robots in the

†Adapted from [1]

environment will result in a *joint* configuration space for the system of n robots described by $\overline{\mathcal{C}} = \mathcal{C} \times \mathcal{C} \times \cdots \times \mathcal{C} - \Delta = \mathcal{C}^n - \Delta$, where we take the product of n copies of the C-spaces corresponding to each robot, and remove from it the *diagonal* that represents collision of the robots (*i.e.*, $\Delta = \{[\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n] \in \mathcal{C}^n \mid \mathbf{p}_i = \mathbf{p}_j \text{ for some } i \neq j\}$). Clearly, the joint configuration space is a $2n$ dimensional manifold.

As a final example, let us consider a *unicycle* model of the point robot [2], which means, in addition to the position (x, y), the robot has an orientation (θ). Thus the configuration space of the robot is now $(\mathbb{R}^2 - \mathcal{O}) \times \mathbb{S}^1$ (a subset of $SE(2)$).

2 Point-set Topology

Set theory is the study of collections of objects. In many cases that collection can be infinite and uncountable. For example, one may talk about the set of all the points on the surface of a sphere. However, set theory does little in establishing relationship among the objects in a set. For example, if we consider the set consisting of the points on a sphere, it's just a collection of points, each of which is distinct and there is no way of telling which point is "connected" to which other point in the set to give the sphere its familiar shape. That's where topology comes to the rescue. A *topology* consists of a set, along with the additional information on "grouping"/"collection" of the objects inside the set. Such "groupings" are called *open subsets* of the set.

Definition 2.1 (Topology [4])**.** A topology on a set X is a collection, T, of subsets of X, containing both X and $\emptyset$, and closed under the operations of intersection and union. Together, the tuple (X, T) is called a *topological space*, and the elements of T are called *open sets* of the topological space.

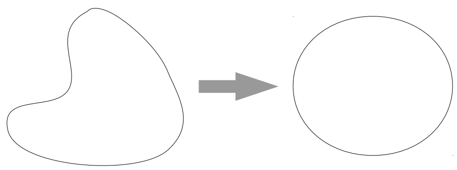
Often, when there is a standard topology, by convention, for a space X, one can refer to the topological space simply as X. One of the most important consequences of defining topology is that we now have the notion of *continuity*.

Definition 2.2 (Continuous Functions Between Topological Spaces [4])**.** Given two topological spaces, (X_1, T_1) and (X_2, T_2), consider a map $f : X_1 \to X_2$. For any subset $U \subseteq X_2$, define $f^{-1}(U) = \{x \in X_1 \mid f(x) \in U\}$ (note that if $f^{-1} : Y \to X$ exists, this definition simply generalizes it to subsets of Y). Then f is said to be continuous if for each open set $V \in T_2$, the set $f^{-1}(V)$ is in T_1 (*i.e.* an open set).

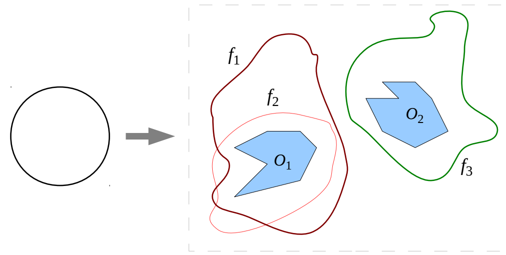
Note that for continuity, f need not map open sets to open sets. That is, for $W \in T_1$, its image $f(W)$ need not be in T_2. When a continuous function is also injective, it is called an *embedding* (embedding of X in Y).

Starting with these basic definitions, one can make assertions on certain properties of the topological space, construct one topological space from another, and establish relationships between them. This is the primary affair of the field of *point-set topology*. However, one can do little algebra or actual computation on a topological space using this. That's where the field of *algebraic topology* gets introduced.

Before we proceed to algebraic topology, we state a few definitions that follow from basic point-set topology.



(a) Homeomorphic spaces (both equivalent to $\mathbb{S}^1$). The apparent difference between the two spaces is due to their embedding in $\mathbb{R}^2$. The spaces themselves are topologically equivalent.



(b) Continuous functions, $f_i : \mathbb{S}^1 \to (\mathbb{R}^2 - O_1 \cup O_2)$, which are also embeddings (injective). f_1 can be continuously deformed to f_2 (they are homotopic), but not to f_3 (f_1 and f_3 are not homotopic).

Figure 2: *Homeomorphic spaces* and *homotopic functions.*

2.1 Homeomorphism, Homotopy, Deformation Retract and Homotopy Equivalence

The first fundamental equivalence relation among topological spaces is that of *homeomorphism.* Topologically, two topological spaces are homeomorphic if they essentially are the same topological space (with, possibly, 'renaming'/'relabeling' of the items in one of the set and its topology to obtain the other). In presence of an embedding, informally, two spaces are homeomorphic if one can be *continuously deformed* into the other without causing cuts or tears in the space (*i.e.* open sets remain open). This is popularly exemplified using a donut and a coffee cup with a handle, and how, to a topologist, they are one and the same. Figure 2(a) shows a more modest example of homeomorphism.

Definition 2.3 (Homeomorphism [4], Fig. 2(a))**.** Two topological spaces X and Y are *homeomorphic* if there exists a bijective function $f : X \to Y$ (which implies the inverse, $f^{-1} : Y \to X$, exists and is bijective) such that both f and f^{-1} are continuous. f (which may not be unique) is called a homeomorphism between the spaces.

A fundamental equivalence relation among continuous functions defined between two fixed topological spaces is *homotopy.* Informally, two functions are homotopic if one can be continuously changed into another.

Definition 2.4 (Homotopy [3], Fig. 2(b))**.** Two continuous function between the same topological spaces, $f_1, f_2 : X \to Y$, are called *homotopic* if there exists a continuous function $F : X \times [0,1] \to Y$ (where, $[0,1]$ is assumed to have the standard *Euclidean topology*, and '$\times$' induces the *product topology* to the product space) such that $F(x,0) = f_1(x)$ and $F(x,1) = f_2(x)$, for all $x \in X$. Concisely we express this relationship as $f_1 \simeq f_2$. The function F (which may not be unique) is call a *homotopy* between f_1 and f_2. Informally, we say that f_1 can be *homotoped* to f_2 and vise-versa.

The idea of *deformation retract* is that given a topological space X, and a subspace A (a subset with *subspace topology* [4]), we ask the question whether or not the space X may be continuously 'shrunk' and 'deformed' to A without causing any 'cut' or 'tear'. If it can, we call A a deformation retract of X (Figure 3). Consider the identity map $\text{id}_X : X \to X$ (Figure 3(a)). Now start 'shrinking' X gradually to 'collapse' on

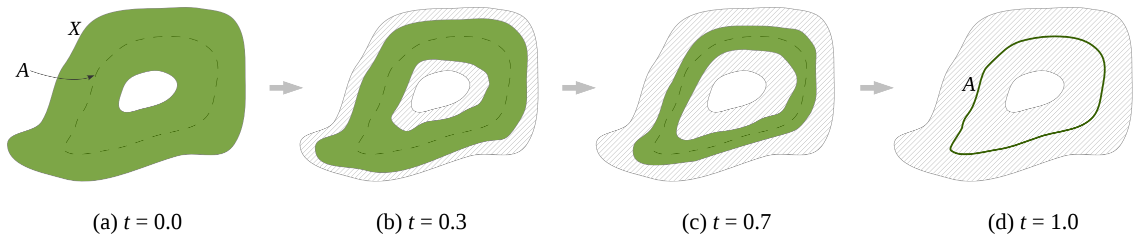


Figure 3: A deformation retraction of X to $A \subseteq X$. For each t, the green area is $F(X, t)$.

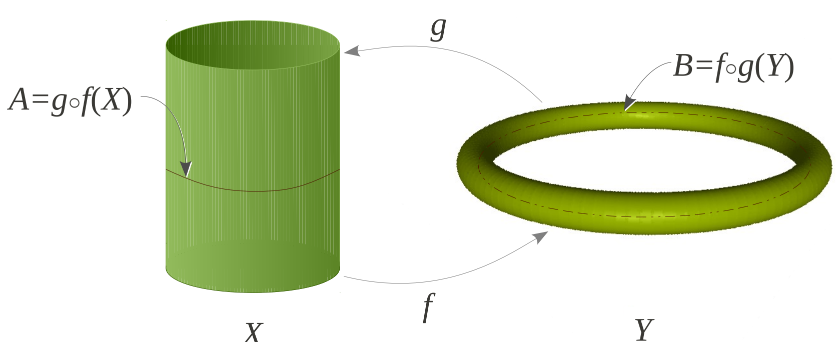


Figure 4: A cylinder (hollow, without lids) and a solid torus are homotopy equivalent. Each of them is homotopy equivalent to a circle.

to A. At every step of the shrinking process what we have is an embedding of X into itself such that the image of the embedding is the 'shrunk' version of X at that step (Figure 3(b,c)). Eventually we 'shrink' X to A (Figure 3(d)).

Definition 2.5 (Deformation Retract [3], Fig. 3)**.** A subspace A (with *subspace topology*) is called a deformation retract of a topological space X if there exists a continuous function $F : X \times [0, 1] \to X$ such that

- $F(x, 0) = x,\ \forall x \in X$ (*i.e.* $F(\cdot, 0) \equiv \mathrm{id}_X$ is the identity map on X),
- $F(a, t) = a,\ \forall a \in A, t \in [0, 1]$, and,
- $F(x, 1) \in A,\ \forall x \in X$.

F (which may not be unique) is called a *deformation retraction* from X to A. Since A is a subspace of X, we can interpret F as a homotopy between the identity map id_X and the map $f_1 \equiv F(\cdot, 1)$ whose image is in A.

It is important to note that $f_1 \equiv F(\cdot, 1) : X \to X$ is homotopic to the identity map on X. Had A not been given beforehand, and instead, we were given a function $f_1 : X \to X$ that is homotopic to id_X, the image of f_1 would clearly be a deformation retract of X.

The fact that A needs to be a subspace of X in the definition of deformation retract essentially implies that there is an embedding $i : A \hookrightarrow X$, called the *inclusion*. However A, as an independent topological space, should not require an embedding in X to be described (*e.g* A in Figure 3 is topologically just a circle $\mathbb{S}^1$). We still should be able to describe a similar relationship between them. That's where a generalization of a deformation retract, called *homotopy equivalence*, comes into the picture.

The idea of homotopy equivalence is that instead of explicitly mentioning a subspace A of X, we look at the continuous functions from X to itself via a second space Y (the final image is of course a subspace of X). We then ask if this combination map is homotopic to the identity map on X. We do the same thing with the role of X and Y reversed. If the answer is 'yes' in both the cases, the spaces X and Y are said to be homotopy equivalents.

Definition 2.6 (Homotopy Equivalence [3], Fig. 4)**.** Two topological spaces X and Y are called *homotopy equivalent* if there exists continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map id_X, and $f \circ g$ is homotopic to the identity map id_Y. The function f (and likewise g) is called a *homotopy equivalence*. X and Y are said to have same *homotopy type*, and informally we say one can be *homotoped* to the other.

If A is a deformation retract of X, then they are of course homotopy equivalents. However the converse is not always true. Out of many ways of determining if two spaces X and Y are homotopy equivalents, one approach is to check if each of them deformation retracts to a subspace that is topologically equivalent (homeomorphic). Then they are homotopy equivalent (Figure 4). The other, more formal, approach is to check if there exists a larger space with embeddings of X and Y into it, such that this larger space deformation retracts to both X and Y.

2.2 Contractible Space

Definition 2.7 (Contractible Space [3])**.** A topological space X is called *contractible* if the identity map on it, id_X, is homotopic to a constant map (a function taking every points in X to a fixed point $x_0 \in X$).

The intuition behind contractibility is that the space can be pulled (contracted) continuously towards a point inside it. It is important to note that a contractible space need not be finite. For example, $\mathbb{R}^D$ is contractible for any finite D. This is because, for any point $p \in \mathbb{R}^D$ one can construct a map $f_p : [0, 1] \to \mathbb{R}^D$ so that $f_p(0) = p$, $f_p(0) = 0$ (the origin), and $F(p, t) = f_p(t)$ is continuous.

References

[1] Subhrajit Bhattacharya. *Topological and Geometric Techniques in Graph-Search Based Robot Planning.* PhD thesis, University of Pennsylvania, January 2012.

[2] B. d'Andrea Novel, G. Campion, and G. Bastin. Control of nonholonomic wheeled mobile robots by state feedback linearization. *The International Journal of Robotics Research*, 14(6), Sept. 1995.

[3] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2001.

[4] James Munkres. *Topology.* Prentice Hall, 1999.