Configuration Spaces and Topology †

Supplementary Notes – Part 2

Introduction to Algebraic Topology

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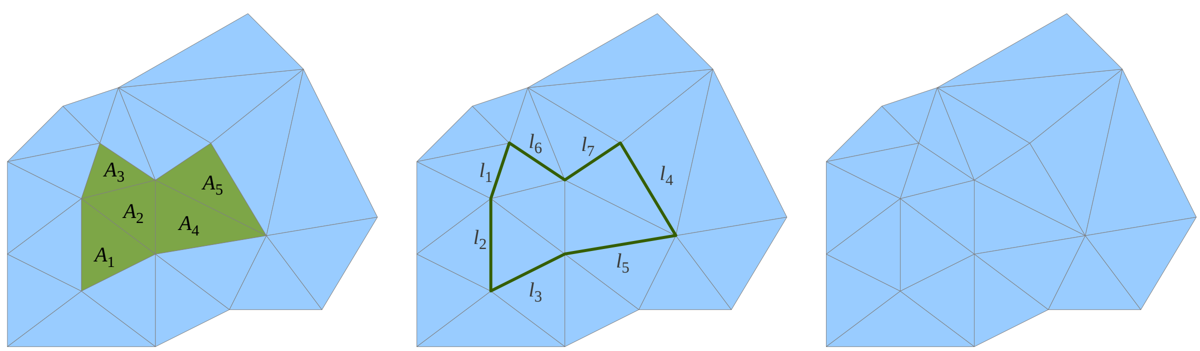
1 Motivation for Algebraic Topology

Algebraic topology imparts certain algebraic (primarily *group*) structures to a topological space, and allows interpretation of the structure of the topological space by analysis of the algebraic structures. In this section we will motivate some basic ideas behind algebraic topology without going into too much technical details. Instead, we will use some simple illustrations to explain them.

The first step in imparting an algebraic structure to a topological space is to describe the space in terms of a sequence of groups (in simple cases, vector spaces, which are themselves groups with additional structures), and maps between them. This algebraic object will be called a *chain complex*. While it is not necessary to discretize/triangulate a topological space to describe a chain complex on it (as we will shortly do), for ease of understanding we make this simplifying discretization. Each discrete element in this discritization is called a simplex (Figure 1) – the vertices will be called 0-simplices, the edges 1-simplices, the triangles 2-simplices, tetrahedrons 3-simplices, etc. Formally, a n-simplex on a topological space, X, is a map from a *standard n-simplex* [4] to X. However, most often, we will informally refer to the image of a n-simplex as the n-simplex itself.



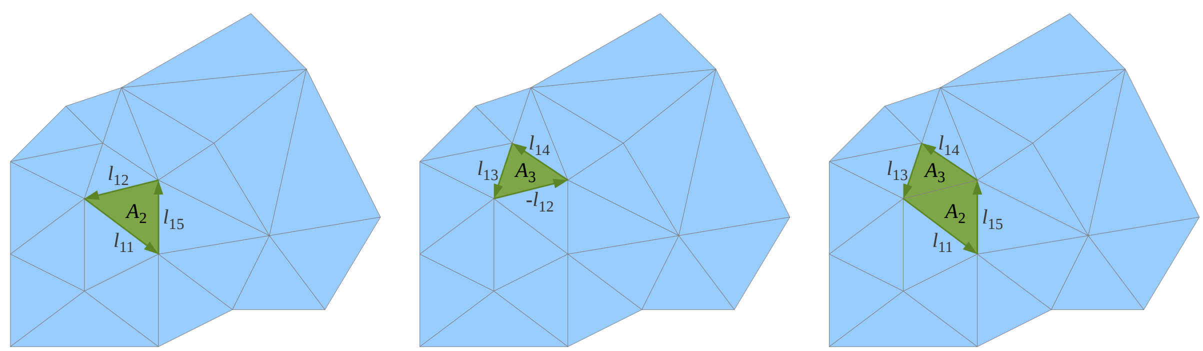
(a) A 2-chain. (b) The boundary of the area (a 1-cycle). (c) The boundary of the boundary is empty.

Figure 1: Boundary operator acting on a chain twice gives an empty chain.

† Adapted from [1]

1.1 Boundary Operator

Consider a patch of area on the plane that is discretized into simplices as in Figure 1. Out of all the 2-simplices (triangles), we pick a few – ones marked by green color as in Figure 1(a). We simply call those triangles $A_1, A_2, \cdots, A_5$ (note that by A_i we do not mean the 'area', but the whole triangle as an abstract object/set). Thus the region they cover is denoted by $A_1+A_2+\cdots+A_5$ (where, for now, we can interpret '+' as an union). Figure 1(b) shows the boundary of the chosen area and is likewise represented by $\sum_{i=1}^{7} l_i$ (each 1-simplex or edge is arbitrarily labeled l_i). However, if we now look at the boundary of $\sum_{i=1}^{7} l_i$, it is clearly empty (in general, such boundary could have been made up of 0-simplices or vertices). This last observation is a key motivation behind constructing a *chain complex*. This observation extends to higher dimensions and any topological space as well. For example, in a 3 dimensional space discretized into tetrahedrons, if we pick a few of those tetrahedrons (3-simplices) to define a volume, and take the boundary of that volume (which will be a closed surface), this boundary will itself have an empty boundary. Thus, boundary of a boundary is always empty. In a naive notation, if $\partial_2(A)$ represents the boundary of an area A, and $\partial_1(l)$ represents the boundary of a curve l, what we just stated can be summarized as $\partial_2(\sum_{j=1}^{5} A_j) = \sum_{i=1}^{7} l_i$, and $\partial_1(\sum_{i=1}^{7} l_i) = 0 = \partial_1 \circ \partial_2(\sum_{j=1}^{5} A_j)$. In general, $\partial_n \circ \partial_{n+1} = 0$.



(a) In the boundary of A_2 we have already labeled the edge l_{12}.

(b) The boundary of A_3 needs to have $-l_{12}$ as an edge. We can re-label it to, say, l_{21}, but then we will need to equate it to $-l_{12}$ to ensure distributivity of ∂_2.

(c) Upon adding the boundaries of A_2 and A_3, the edges l_{12} and $-l_{12}$ cancel out, and we obtain the boundary of $A_2 + A_3$.

Figure 2: Distributivity of Boundary Operator.

1.2 Distributivity of Boundary Operator and Orientation

One of the properties that we would like the boundary operator, ∂_n, to have is distributivity. That is, for example, we would like to be able to write $\partial_2(A_i+A_j) = \partial_2(A_i)+\partial_2(A_j)$. This will let us assert that boundary of boundary is empty, purely from algebraic conditions, without looking at a picture: $\partial_1 \circ \partial_2(\sum_i A_i) = \sum_i \partial_1 \circ \partial_2(A_i) = \sum_i 0 = 0$ (since boundary of boundary of an individual triangle is always empty). This would enable us develop a linear algebra. This requires that we assign some sign (directionality) to each of l_i. Consider a single 2-simplex, A_2, as shown in Figure 2(a). Its boundary is $l_{11} + l_{15} + l_{12}$. Now consider the 2-simplex A_3 (Figure 2(b)). Since A_2 and A_3 share the common edge, l_{12}, which will lie inside $A_2 + A_3$, we need to make sure that somehow this edge gets canceled out when we add $\partial_2(A_2)$ to

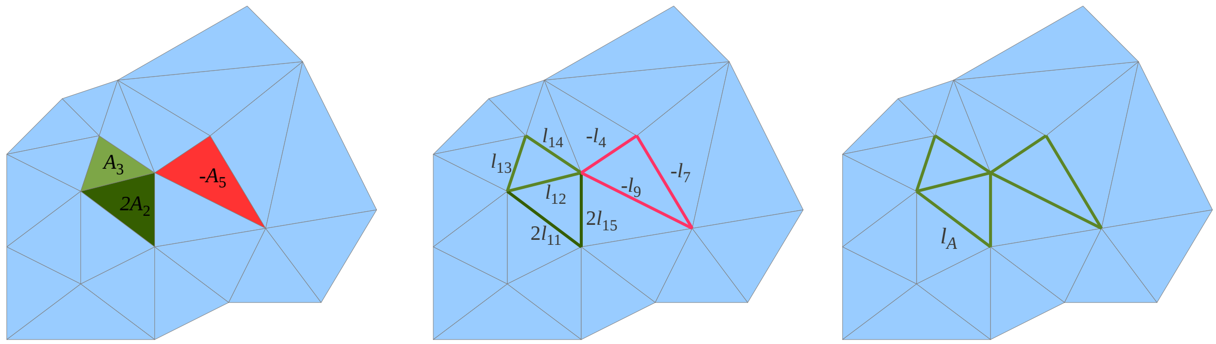
$\partial_2(A_3)$ to obtain $\partial_2(A_2 + A_3)$ (Figure 2(c)). This is attained by giving a directionality of every segment l_i, represented as $\pm l_i$, and noting that $l_i + (-l_i) = 0$. There is nothing special about the dimension 1 of the 1-simplices, and we can in fact assign directionality to simplices of every dimension (vertices, edges, triangles, tertahedrons, etc.). The definition of direction has to be such that it admits distributivity of the boundary operators ∂_n. For example, if one considers $-A_2$ in Figure 2(a), in order to be consistent with the fact that $\partial_2(A_2 + (-A_2)) = \partial_2(A_2) + \partial_2(-A_2) = 0$, we need to have the boundary of $-A_2$ (*i.e.* $\partial_2(-A_2)$) as $(-l_{11}) + (-l_{15}) + (-l_{12})$, that is the original line segments with reverse orientation.

1.3 Group Construction

By now it is easy to see a group structure emerging. For example, for every line segment l_i we have defined an inverse element, $-l_i$ so that they add up to 0, the identity element. Also, we have developed the intuition of how the binary operator '+' works between l_i and l_j for $i \neq j$ or $l_j = -l_i$. All that we now need to do to make the set of possible combinations of the 1-simplices (*e.g.* $l_{i_1} + l_{i_2} + \cdots$ is an arbitrary combination – called a 1-chain) an algebraic group is to *close* it under the operation of addition. Earlier we have related $l_i + l_j$ with taking union of the line segments l_i and l_j. However, if we write $l_i + l_i$, an interpretation in terms of union, will simply mean l_i. This will not be consistent with our attempt to define a group. Thus, we define a new element $2l_i := l_i + l_i$. This can be interpreted as taking the line segment two times on top of itself (similar to *disjoint union*). However it is to be kept in mind that this is absolutely an algebraic construction. Following along similar lines, we can define $2l_i, 3l_i, 4l_i, \cdots$, and the corresponding inverses $-2l_i, -3l_i, -4l_i, \cdots$. In general $nl_i := l_i + l_i + \cdots (n \text{ times})$, and $-ml_j := (-l_j) + (-l_j) + \cdots (m \text{ times})$. Thus we have constructed an abelian group that is *freely generated* by $l_1, l_2, l_3, \cdots$. We represent this group by $C_1(X)$ (where, X is the topological space which we discretized to create the simplices), where the subscript 1 refers to the dimension of the simplices. Of course we can do similar treatment for simplices of all dimensions (*e.g.* Figure 3). The group for the n-dimensional simplices is written as $C_n(X)$.

1.4 Coefficients

One further generalization of the said group construction is that with arbitrary coefficients. The idea can be described as follows: So far we have constructed elements like $2l_i, 3l_i, \cdots$ (*i.e.* l_i with integer coefficients). However, very often, one may come across problems where non-integer coefficients arise very naturally. One typical inspiration comes from an electrical network that can be modeled as a simplicial 1-complex (the coefficients on the 1-simplices represent the currents passing through them, and the coefficients on the 0-simplices represent the voltage at the nodes). Then, because of Kirchhoff's law, the sum of the incoming currents at any node is equal to the sum of the outgoing currents at the node. Consequently, any closed loop of current represents a 1-cycle in the complex [3]. However, note that now we need to define coefficients over $\mathbb{R}$ since the currents can assume any real number (*e.g.* $2.56\ l_i$). Moreover, due to the *superposition theorem for electrical circuits* [2], one can add currents, and hence add the 1-cycles. For such additions we naturally borrow the definition of additions from the real numbers, $\mathbb{R}$ (which is a group under '+' operator with 0 as identity element), and the additions happen simply on the coefficients of l_i. In fact we can generalize the coefficients to arbitrary algebraic structures (like *groups*, *rings*, *fields*, etc.). Thus, if G is an algebraic structure and an abelian group under an addition operation, '+', then we define $C_1(X; G)$ (as a generalization of $C_1(X)$) to be abelian group in which every element is represented by an ordered set of



(a) A graphical representation of $A_3 + 2A_2 + (-A_5) \in C_2(X)$. Colors represent the integer coefficients – green positive, red negative, darker is higher. Note how the coefficient for the other 2-simplices is 0 represented by light blue. This type of arbitrary combination is called a *chain* (a 2-chain in this case). Thus, one can interpret this as a order set of coefficients, $[0, 2, 1, 0, -1, 0, 0, 0, \cdots]$, for the corresponding 2-simplices.

(b) The boundary of the 2-chain shown in (a). The boundary is $l_{13} + l_{14} + l_{12} + 2l_{11} + 2l_{15} - l_4 - l_9 - l_7$. Note that we no more use an 'arrow' to represent the direction of the 1-simplices. Since we can now have arbitrary integer coefficients, color representation of the coefficients is the preferred way of visualization. Green indicates positive, red indicates negative. Note how a $-l_{12}$ from $\partial_2(A_3)$ adds to $2l_{12}$ from $\partial(A_2)$ to give the l_{12} in the figure.

(c) However, one can set $l_A = l_{13} + l_{14} + l_{12} + 2l_{11} + 2l_{15} - l_4 - l_9 - l_7$, and considered to be a generating element of $C_1(X)$ (by change of basis). Then we will be using light green to represent l_A with coefficient 1. Then, for example, $l_A + l_{12}$ will be represented by this same figure as above, except with the edge corresponding to l_{12} being marked with darker green.

Figure 3: A 2-chain and its boundary, with coefficients (which includes direction information) represented by colors.

coefficients $[g_1, g_2, g_3, \cdots]$, $g_i \in G$ (these are the coefficient of $l_1, l_2, l_3, \cdots$), that inherits the operator '+' from G by the element-wise operation $[g_1, g_2, g_3, \cdots] + [g'_1, g'_2, g'_3, \cdots] = [g_1 + g'_1, g_2 + g'_2, g_3 + g'_3, \cdots]$. It is to be noted that the addition operator is a chosen preferred operator of G. The inheritance of other operators from G to $C_1(X; G)$ are subject to independent definitions. Of course, once again, this is general for arbitrary dimensions, and for n-dimensional simplices we have the algebraic structures $C_n(X; G)$.

The boundary operator, which was distributive, is extended to being *linear* (informally. More precisely it is a group homomorphism) when we have the coefficients (this is mostly by definition than anything else – by extending the definition of boundary operator to chains with coefficients). Thus, from our previous example, if $A = g_a A_2 + g_b A_3 \in C_2(X; G)$ for $g_a, g_b \in G$, we have $\partial_2(A) = g_a \partial_2(A_2) + g_b \partial_2(A_3) = g_a(l_{11} + l_{15} + l_{12}) + g_b(l_{13} - l_{12} + l_{14}) = g_a(l_{11} + l_{15}) + g_b(l_{13} + l_{14}) + (g_a - g_b)l_{12}$. Substituting $g_a = g_b = 1$ for $G = \mathbb{Z}$, we obtain our previous result. Also, by linearity, $\partial_1 \circ \partial_2(A) = g_a \partial_1 \circ \partial_2(A_2) + g_b \partial_1 \circ \partial_2(A_3) = g_a 0 + g_b 0 = 0$. In general $\partial_n \circ \partial_{n+1} = 0$ due to linearity and the fact that for a $(n+1)$-simplex σ^{n+1}, $\partial_n \circ \partial_{n+1} \sigma^{n+1} = 0$. More precisely, the boundary operator is a *group homomorphism*.

1.5 Vector Space

One can very well compare $C_n(X)$ to a vector space (especially when the coefficients are in field $\mathbb{R}$). For example, in Figure 3, each l_i may be thought to be an basis vector forming a basis set in a N-dimensional vector space (where N is the total number of 'edges' or 1-simplices in the discretization of X). Thus, any linear combination of the basis vectors, $\sigma = a_1 l_1 + a_2 l_2 + \cdots + a_N l_N$, will represent some 1-chain (Figure 3(b)).

Then, $C_n(X)$ is very much like a vector space spanned by those basis vectors. One can even talk about change of basis (Figure 3(c)). In particular, if the coefficients are in a field, then $C_n(X)$ is indeed a vector space. If elements from this vector space can be represented by coefficient vectors $[a_1, a_2, a_3, \cdots, a_N]^T$ (as described earlier, and assuming there are N numbers of n-simplices in the representation of X), and the elements of the vector space $C_{n-1}(X)$ is represented by coefficient vectors $[b_1, b_2, b_3, \cdots, b_M]^T$ (where we assume that there are M numbers of $(n-1)$-simplices in the finitely discretized representation of X), then the boundary operator ∂_n may be represented by a $M \times N$ matrix. Thus, when a N-dimensional coefficient vector representing a vector from from $C_n(X)$ is left-multiplied by this matrix, we obtain a M-dimensional coefficient vector representing a vector in $C_{n-1}(X)$. Then the kernel and image of ∂_n have simple interpretations borrowing the corresponding concepts from linear algebra.

2 Formal Description of Homology

Once we have established the motivation behind defining chain complexes, we can formally define it in the most general way as follows.

Definition 2.1 (Chain Complex [4], Fig. 4)**.** A *chain complex* is a sequence of abelian groups, $\cdots,\ C_3,\ C_2,\ C_1,\ C_0,\ C_{-1}, \cdots$, along with homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$ for all $n = \cdots, 3, 2, 1, 0, -1, \cdots$. It is commonly represented using the following diagram:

$$\cdots \longrightarrow C_{n+3} \xrightarrow{\partial_{n+3}} C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

with, $\partial_{n-1} \circ \partial_n = 0,\ \forall n$.

Note that in general, chain complexes need not be related to any topological space, as in the independent definition stated above. It is simply a sequence of abelian groups $C_\bullet$ with the operators $\partial_\bullet$ (by subscript '$\bullet$' we informally mean the collection for all n). Such independent studies of chain complex without any reference to topology is known as *homological algebra* and is a field of study by its own right. Algebraic topology borrows significant amount of tools from that field.

In algebraic topology, these groups C_n obviously corresponds to the groups freely generated by the n-simplices on the topological space X, and are written as $C_n(X)$. Chain complexes generated by a finite number of simplices as described earlier are known as *simplicial complex*. However there are other, and more general forms of chain complex that one can define on a topological space – Δ-complex, singular complex, cellular complex, cubical complex, etc.

Since $\partial_n \circ \partial_{n+1} = 0$, we have (see Figure 4)

$$\mathit{Img}(\partial_{n+1}) \subseteq \mathit{Ker}(\partial_n) \subseteq C_n(X)$$

Definition 2.2 (Subgroup of Boundaries [4])**.** $B_n(X) := \mathit{Img}(\partial_{n+1}) \subseteq C_n(X)$ is called the group of n-boundaries (a subgroup of C_n), and is the image of the whole of $C_{n+1}(X)$ under the action of ∂_{n+1}. Elements in $B_n(X)$ (called n-boundaries) are n-chains, each of which are boundaries of some $(n+1)$-chains in X (Figure 5(c)).

Definition 2.3 (Subgroup of Cycles [4])**.** $Z_n(X) := \mathit{Ker}(\partial_n) \subseteq C_n(X)$ is called the group of n-cycles (a subgroup of C_n), and is the *kernel* of ∂_n (*i.e.* all the elements in C_n that maps to the identity element

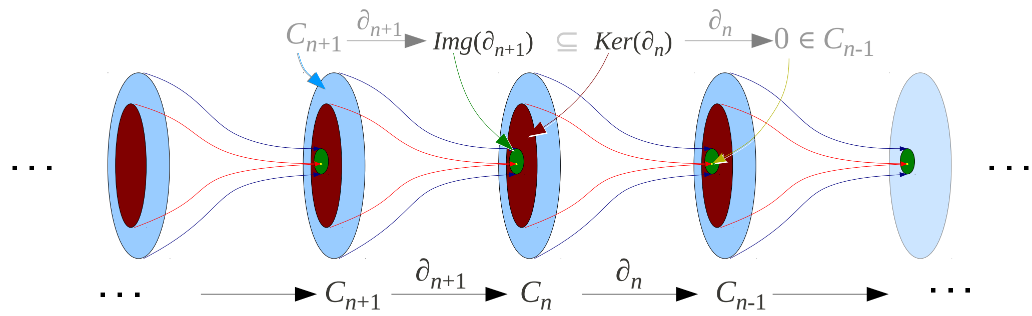


Figure 4: A schematic representation of a chain complex (see [3]). It consists of the sequence of groups, $C_\bullet$, along with homomorphisms $\partial_\bullet$, with the property that $\partial_n \circ \partial_{n+1}(\sigma) = 0$ for any $\sigma \in C_{n+1}$.

in $C_{n-1}(X)$ under the action of ∂_n). Elements in $Z_n(X)$ (called n-cycles) are n-chains that have empty boundary under the action of ∂_n (Figure 5(b)). Of course all n-boundaries are n-cycles as well, but the converse is not true.

It is easy to see that both $Z_n(X)$ and $B_n(X)$ are closed under addition and also the inverse of elements in each of them belong to the sets themselves. Thus they are subgroups.

As discussed earlier, an element $\sigma \in C_n(X)$ is an arbitrary linear combination of n-simplices on X (Figure 5(a)). However, a subset of those will be such that the boundary operator, ∂_n, acts on them to give zero (*i.e.* they are in $Ker(\partial_n)$). These are elements $z \in Z_n(X) \subseteq C_n(X)$ (Figure 5(b)). Furthermore, some out of those are such that they themselves are boundaries of some one higher dimensional chain (*i.e.* they are in $Img(\partial_{n+1})$). Those are elements $b \in B_n(X) \subseteq Z_n(X)$ (Figure 5(c)).

Since, by definition, for any $b \in B_n(X)$ we can find a $\omega \in C_{n+1}(X)$ such that $\partial_n(b) = \partial_n \circ \partial_{n+1}(\omega) = 0$ (by definition of chain complex), we have $B_n(X)$ a subgroup of $Z_n(X)$. Thus, one can now construct the following *quotient group*,

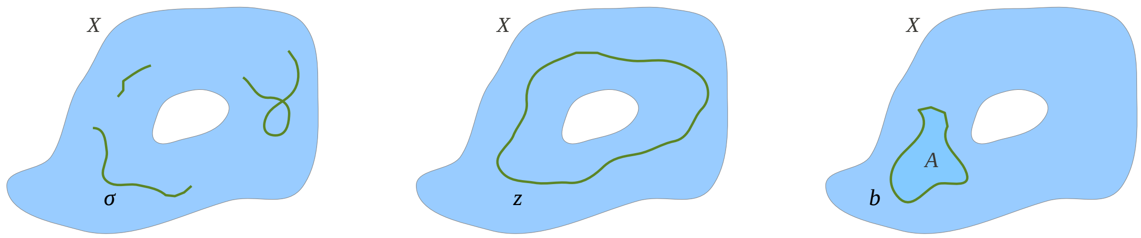
Definition 2.4 (Homology Groups [4])**.** We define the n^{th} homology group of X as ,

$$H_n(X) \;=\; Z_n(X) \;/\; B_n(X)$$

The intuitive description of $H_n(X)$ is as follows: We look at two n-cycles $z_1, z_2 \in Z_n(X)$ (Figure 6). If their difference is boundary of some one-dimensional higher chain (*i.e.* $z_1 - z_2 \in B_n(X)$), we say that they belong to the same *homology class* or are *homologous* (Figure 6(a) and 6(d)), otherwise we say that they are in different *homology classes* (Figure 6(b) and 6(c)). Thus, we have several different homology classes of n-cycles. $H_n(X)$ essentially is the set of all those homology classes. This can be seen clearly by the definition of *group quotient*. Each element of $H_n(X)$ is a partition of $Z_n(X)$ such within each partitions (*i.e.* a homology class) two elements z_1 and $\overline{z}_1$ can be related as $\overline{z}_1 = z_1 + b$ for some $b \in B_n(X)$.

2.1 Group Structure of $H_n(X)$

For a given $z \in Z_n(X)$, we write $[z] \in H_n(X)$ for the homology class of Z. The addition operator of $H_n(X)$ is inherited from $Z_n(X)$ in a rather natural way – we define $[z_1] + [z_2] = [z_1 + z_2]$, where the first addition is the one in $H_n(X)$ (one we are defining), while the addition on the right is well-known for $Z_n(X)$. Also,



(a) A 1-chain. The boundary of this chain is not empty since the curves have end-points (0-chains). This is an arbitrary element of $C_1(X)$.

(b) A 1-chain without a boundary is a 1-cycle. This is an element of $Z_1(X)$.

(c) A chain which is boundary of a 2-chain (the one marked by A). This is an element of $B_1(X)$.

Figure 5: Illustration of *chain*, *cycle* and *boundary*. The topological space X is shown in blue. The figures do not show a discretization explicitly, however one can think it to be discretized into very small simplices (much like what we had in Figure 2 or 3) in order to accommodate almost arbitrary-shaped chains. Every 1-chain shown in this figure are made of 1-simplices, being labeled for the first time, with *coefficients as* 1 (indicated by the color light green – refer to Figure 3(b)).

the identity element (or 'zero') of $H_n(X)$ is the homology class of the boundaries (elements of $B_N(X)$). This can be observed as follows: If z_1 and $\overline{z}_1$ belong to the same homology class (*i.e.* $\overline{z}_1 = z_1 + b$ for some $b \in B_n(X)$), then by definition, $[\overline{z}_1] = [z_1] \implies [z_1 + b] = [z_1] \implies [z_1] + [b] = [z_1] \implies [b] = 0$.

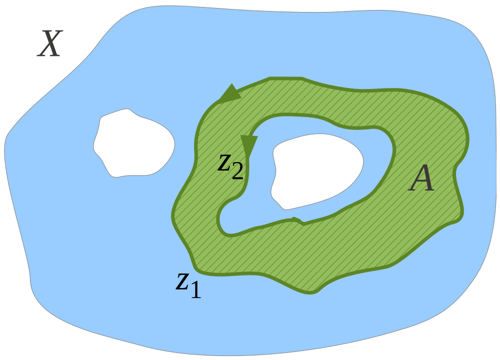
Resorting briefly to our earlier comparison of $C_n(X)$ with a vector space, we can see that $Z_n(X)$ is like a vector subspace (a vector space by its own right). Thus, $B_n(X)$ is a subspace of $Z_n(X)$. Then $H_n(X)$ may be thought of as the vector subspace of $Z_n(X)$ which is orthogonal to $B_n(X)$ such that $H_n(X)$ and $B_n(X)$ spans the whole of $Z_n(X)$.

2.2 Relative Homology

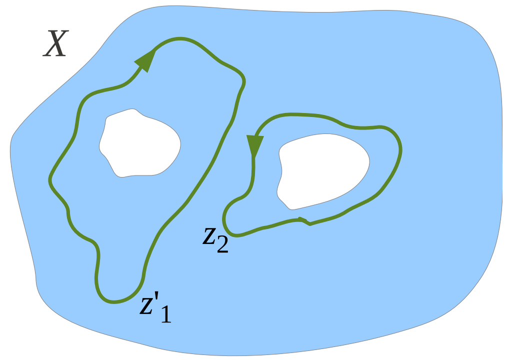
Given $C_\bullet(X)$, a chain complex on X, and a subspace S of X (S and X are together written as (X, S) and is called a *pair of spaces*), one can construct the *subcomplex* $C_\bullet(S)$, where each $C_n(S)$ is a subgroup of $C_n(X)$ freely generated by the n-simplices that fall inside S. Then one can talk about quotient groups $C_n(X)/C_n(S)$. These quotient groups are written as $C_n(X, S)$ for brevity. Thus, there is a quotient map $j : C_n(X) \to C_n(X, S)$ such that given any n-chain $\sigma \in C_n(X)$, if we *trivialize* the part of the chain that lies inside S, we obtain $j(\sigma) \in C_n(X, S)$. It is analogous to taking projection of σ on the subspace orthogonal to the subspace $C_n(S)$. One can then extend the boundary operator ∂_n quite naturally to define $\partial_n : C_n(X, S) \to C_{n-1}(X, S)$. Then it is not difficult to see that $C_\bullet(X, S)$ form a chain complex,

$$\cdots \longrightarrow C_{n+1}(X, S) \xrightarrow{\partial_{n+1}} C_n(X, S) \xrightarrow{\partial_n} C_{n-1}(X, S) \xrightarrow{\partial_{n-1}} \cdots$$

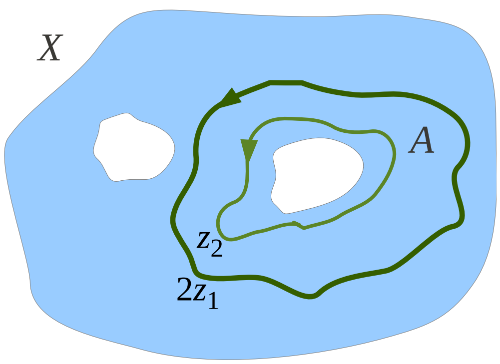
with $\partial_{n-1} \circ \partial_n = 0$, $\forall n$. The image of a n-chain $\sigma \in C_n(X)$ under the action of j is typically called a *relative chain* and is an element of $C_n(X, S)$. We can compute homology groups for the above chain complex. Those are called *relative homology* groups, and are represented by $H_n(X, S)$. It is to be noted that these, in general, are not same as $H_n(X)/H_n(S)$. Any $\sigma \in C_n(X)$ lying inside completely S will be a trivial relative chain



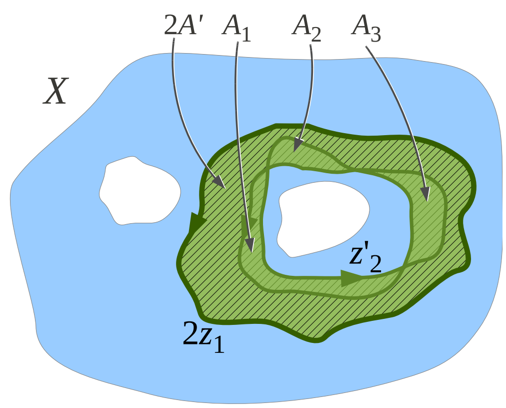
(a) z_1 and z_2 are 1-cycles that are in the **same** *homology class*. This is because z_1 and $-z_2$ (z_2 with reversed orientation) forms the boundary of a 2-chain, A, (one consisting of all the 2-simplices in the annular region with unit coefficients). That is, $z_1 - z_2 = \partial_2(A) \in B_1(X)$



(b) z'_1 and z_2 are 1-cycles that are in **different** homology classes. This is because one cannot find a 2-cycle A such that $z_1 - z_2 = \partial_2(A) \in B_n(X)$



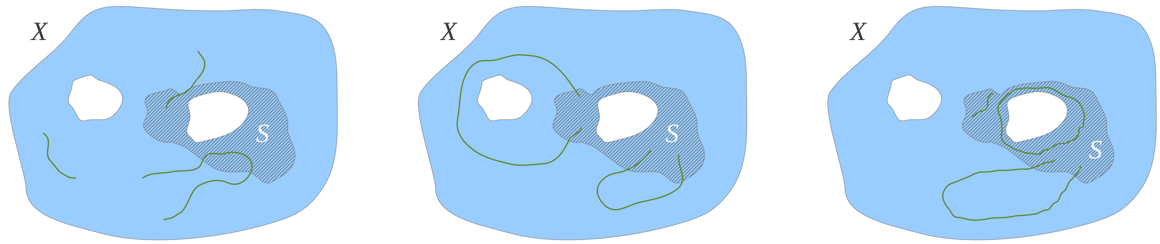
(c) $2z_1$ (where z_1 is the same as in (a)) and z_2 are 1-cycles that are in **different** homology classes. The coefficient 2 for z_1 (which may be thought of as 2 copies of z_1 placed on top of one another) is indicated by the darker color (refer to Figure 3(b)).



(d) $2z_1$ and z'_2 are 1-cycles that are in **same** homology classes. This is because we can write $2z_1 - z_2 = \partial_2(2A' - A_1 - A_2 - A_3) \in B_1(X)$.

Figure 6: Cycles in same and different homology classes. Discretization and color coding of coefficients similar to before (Figure 5).

$j(\sigma) = 0 \in C_n(X, S)$ and is a *relative boundary* (Figure 7(c)). And any n-chain $\sigma \in C_n(X)$ with boundary lying completely inside S will be a *relative n-cycle* $j(\sigma) \in Z_n(X, S)$ (Figure 7(b)).



(a) Relative chains. The 1-chains in X have some boundary (end point, 0-chain) lying outside S.

(b) Relative cycles. The 1-chains in X have boundaries lying inside S. Thus in the relative form (*i.e.* their image under quotient map j), they have no boundary. Hence these are relative cycles.

(c) Relative boundaries. For these 1-chains in X, either the whole of it lies inside S, thus making it trivial, or they are boundaries of some relative 2-chain.

Figure 7: Relative chains on $C_1(X, S)$, and chains that are relative cycles and relative boundaries.

3 Properties of Homology

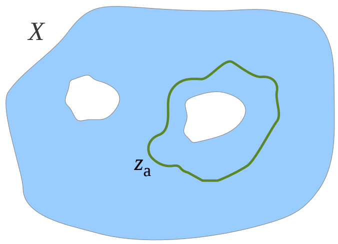
In this section we will mostly state some results and explain their implications, but without detailed proofs. The reader may refer to [4] for the proofs and detailed discussion.

3.1 Interpretation of Homology Groups

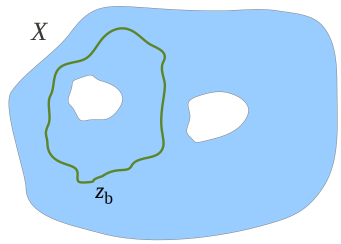
Each element of the n^{th} homology group, $H_n(X)$, as we just saw, represents a class of n-cycles that differ by n-boundaries. There is however an even more intuitive and useful interpretation of the homology groups – the rank (or informally, the dimensionality) of the group tells us about the n^{th} Betti number (informally, the number of $(n+1)$-dimensional 'holes' when $n > 0$, and number of connected components when $n = 0$) of the topological space X. This is not difficult to see from the example in Figure 6. One can see that corresponding to the two holes in the space X, there are two distinct types of cycles that are not boundary (called non-trivial cycles) – one that goes around the right hole (Figure 8(a)), and other that goes around the left hole (Figure 8(b)). These are the generators of $H_1(X)$. In fact a direct computation of $H_1(X)$ reveals that it is isomorphic to the group $\mathbb{Z}^2$ (*direct sum* of two copies of the integers' group, which is a group under addition) – thus, the first Betti number of the space, b_1, is 2. The homology class of any other cycle in the space can be expressed as a linear combination of these two homology classes (Figure 8(c)).

3.2 Indifference to Method of Construction of Chain Complex

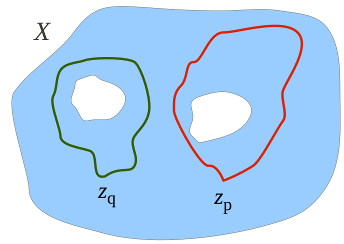
The homology groups of a topological space are indifferent to the method of construction of the chain complex used to compute the homology groups. The intuitive idea is that given a topological space X, one can create a chain complex in many different ways on it. Even a finite simplicial discretization (*e.g.* Figure 3) can be created in infinite variety. Besides, there are other types of chain complexes that can be constructed on a topological space (like Δ-complex, singular complex, cellular complex, cubical complex – see [4] for details). Although these create vastly different chain complexes, $\{C_\bullet(X), \partial_\bullet\}$, the homology groups $H_n(X)$ computed using any of them will however be the same as long as we stick to the same coefficient group. This



(a) $[z_a]$ is a generator of $H_1(X)$.



(b) $[z_b]$ is another generator of $H_1(X)$.



(c) $z = az_p + bz_q \in Z_1(X)$ is an arbitrary 1-cycle. Its homology class however is $[z] = a[z_p] + b[z_q] = a[z_a] + b[z_b]$, *i.e.* can be expressed as a linear combination of the generating homology classes.

Figure 8: The rank of homology group gives the Betti number. The homology class of an arbitrary cycle can be expressed as the linear combination of the generators of the homology group.

may be intuitive from the discussion in the previous paragraph, where we saw that the homology groups provide information about the Betti numbers of X – a topological invariant of X. However a rigorous proof of this fact is quite elaborate and involved [4].

3.3 Functoriality

Homology is a functor from the *category* of topological spaces to the category of groups [4]. The simple meaning of this statement is that if there are two topological spaces X and Y, and if there is a continuous function $f : X \to Y$, then there exists a group homomorphism $f_{*:n} : H_n(X) \to H_n(Y)$, $\forall n$ such that for a cycle $z \in Z_n(X)$ the following holds: $f_{*:n}([z]_X) = [f(z)]_Y$. Here, by $f(z)$ we mean the image of z in Y under the action of f (which will still be a cycle), and by the subscripts X and Y of the square brackets we mean the homology class of the cycle in the respective topological space (*i.e.* elements of $H_n(X)$ or $H_n(Y)$ respectively). We say that the map f *induces* the homomorphisms f_* between the homology groups (where, by the subscript '$*$' we mean the collection of all the induced homomorphisms $\cdots, f_{*:2}, f_{*:1}, f_{*:0}$). Also, due to functionality, if there are maps $f : X \to Y$ and $g : Y \to Z$, then $(g \circ f)_* = g_* \circ f_*$.

3.4 Homotopy Invariance

If two spaces X and Y are *homotopy equivalent* (Definition **??**), then their homology groups are isomorphic (*i.e.* they essentially are the same groups). In notation, $H_n(X) \approxeq H_n(Y)$, $\forall n$. This is an important result in algebraic topology.

However, if we know that the n^{th} homology groups of two spaces are isomorphic (*i.e.* $H_n(X) \approxeq H_n(Y)$), it is often nontrivial to find a map $f : X \to Y$ that *induces* the isomorphism. For example we can have $H_n(X) \approxeq H_n(Y) \approxeq \mathbb{Z}^p$, but if $f : X \to Y$ is such that every point on X is mapped to a single point $y_0 \in Y$ (*i.e.* a constant map), then the map $f_{*:n}$ is a zero map (which is still a homomorphism, but not an isomorphism). On the other hand, if f is a *homotopy equivalence* between X and Y, then f_* are isomorphisms.

3.5 Long Exact Sequence (LES)

A long exact sequence is a special type of chain complex consisting of sequence of abelian groups, $A_\bullet$, and chain maps between the groups, $p_\bullet$, with the property that $Img(p_{n+1}) = Ker(p_n),\ \forall n$ (instead of just being subset as it was the case for chain complex). Thus long exact sequences are obviously chain complexes as well. The sequence can be finite or infinite as in the case of chain complex.

An important result in algebraic topology is that given a pair of spaces, (X, S), the following sequence is a LES:

$$\cdots \longrightarrow H_n(S) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,S) \xrightarrow{\partial_*} H_{n-1}(S) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{j_*} \cdots$$

where, by the subscripts '$*$' we mean the corresponding induced homomorphism to be used for appropriate value of n. i_* is the homomorphism induced (due to *functoriality* of homology) by the inclusion map $i : S \hookrightarrow X$. j_* is induced by the quotient map at the chain level (as opposed to the topological space) $j : C_n(X) \to C_n(X)/C_n(S)$. And ∂_* is a homomorphism that maps the homology class of relative cycles in (X, S) to the homology class of its boundary in S. A more detailed discussion on properties of LES can be found in pp. 114 of [4].

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