Supplementary Notes on Platonic Solids and Distances

Math 170: Ideas in Mathematics

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Platonic Solids

A Platonic Solid or a *regular solid* is a polyhedron with all its faces being the same regular polygon. There are exactly 5 platonic solids:

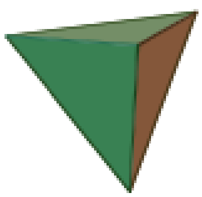
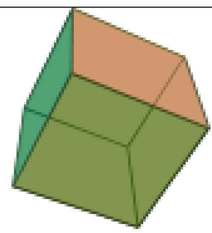
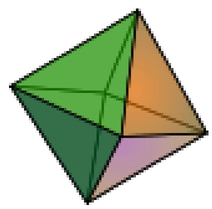
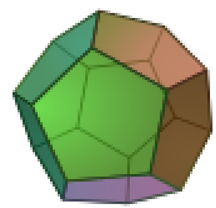
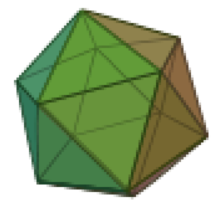
 **Tetrahedron**	 **Cube**	 **Octahedron**	 **Dodecahedron**	 **Icosahedron**
$F=4$ $V=4$ $E=6$	$F=6$ $V=8$ $E=12$	$F=8$ $V=6$ $E=12$	$F=12$ $V=20$ $E=30$	$F=20$ $V=12$ $E=30$
$s=3$ $d=3$	$s=4$ $d=3$	$s=3$ $d=4$	$s=5$ $d=3$	$s=3$ $d=5$
Self-dual	Dual to Octahedron	Dual to Cube	Dual to Icosahedron	Dual to Dodecahedron

Figure 1: The five platonic solids and their properties (Image courtesy of Wikimedia Commons).

where, F, E and V are the number of faces, edges and vertices of the Platonic solids, s is the number of sides of each face, and d is the degree of each vertex (*i.e.*, the number of edges emanating from each vertex).

Note that the Platonic solids are all topologically equivalent to a 2-dimensional sphere.

Euler Characteristic Theorem

The *Euler Characteristic* for a platonic solid (or any solid as a matter of fact) is the number $V-E+F$. The Euler Characteristic Theorem says that for any solid object that is topologically equivalent to a 2-dimensional sphere, $V-E+F=2$. In particular, the Euler Characteristic of a Platonic solid is $V-E+F=2$.

Classification of Platonic Solids

We prove that there are exactly five platonic solids. That is there cannot be another solid other than the five mentioned ones which has the same regular polygon as all its faces.
The following equations play key role in this proof:

We first notice that each vertex is connected to d edges (d being the degree of a vertex) and each edge is shared by two vertices. Thus,

$$V\ d = 2E \tag{1}$$

Likewise, each face has s sides, and each edge is shared by 2 faces. Thus,

$$F\ s = 2E \tag{2}$$

Using the above two equations and plugging V and F in terms of E, d and s in the Euler characteristic equation we get,

$$\frac{2}{d} + \frac{2}{s} = 1 + \frac{2}{E} \tag{3}$$

The above equation gives us $\frac{2}{d} + \frac{2}{s} > 1$. Also, we note that $s, d \geq 3$ for obtaining a meaningful solid. These together (using a proof by contradiction) also gives us $s, d < 6$.

Then the proof requires us to enumerate all the possible pairs for the values of s and d, *i.e.*, $s, d = 3, 4, 5$, and check which out of these pairs satisfy the condition $\frac{2}{d} + \frac{2}{s} > 1$. Each valid pair (five of them) gives us a Platonic solid.

Refer to class lecture notes for more details and the complete proof.

Developed Representation of Platonic Solids

A developed representation of a platonic solid is obtained by cutting the solid along certain edges so that it can be drawn on a flat plane. A developed representation of each of the 5 Platonic solids is shown in the table below. Although arrows indicating the gluing of the edges that would give us back the Platonic solid has not been shown, that information forms part of the developed representation. **Refer to class lecture notes for** more details. Gluing along specific edges with the same arrows give us back the original Platonic solid.

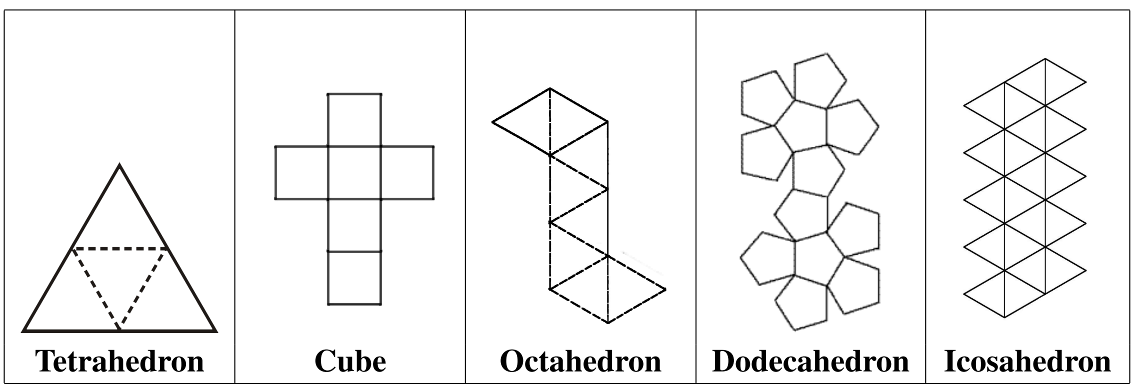


Figure 2: The developed representation of the five platonic solids.

Distances Between Points on a Platonic Solid

Given two points on the surface of a Platonic solid, one is faced with the problem of measuring distance between them. In this context, there are two distinct notions of distance:

Euclidean Distance: *Euclidean Distance* between two points is the length of the shortest line segment connecting the points. One can use Pythagoras' theorem to compute Euclidean distance or make use of coordinate geometry. If in some orthogonal coordinate system the coordinates of two points, p_1 and p_2 are respectively (x_1, y_1, z_1) and (x_2, y_2, z_2), the the Euclidean distance between the points is

$$d_E(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

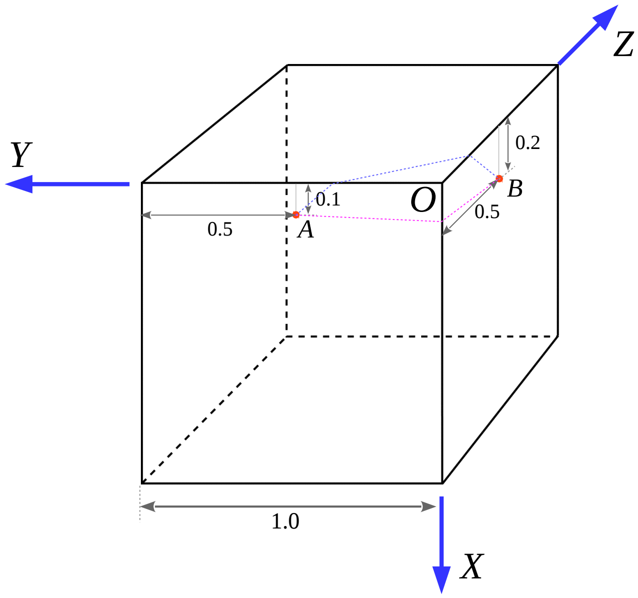
For example, for the unit cube shown in Figure 3(a), for computing $d_E(A,B)$, one can set up the coordinate axes as shown. Using this coordinate system, the coordinates of the points are $A \equiv (0.1, 0.5, 0.0)$ and $B \equiv (0.2, 0.0, 0.5)$. Thus, $d_E(A,B) = \sqrt{(0.1-0.2)^2 + 0.5^2 + (-0.5)^2} = \sqrt{0.51}$. **Refer to class lecture notes for** more details and examples.

Geodesic Distance: Geodesic distance between two points is the length of the *shortest path* connecting the points on the surface of the solid object. Thus, if an ant is navigating on the surface of the solid, that will be the path of it choice to travel from one point to the other. The method for finding geodesic distance between two points on the surface of a platonic solid is to use a developed representation in which the points can be joined by a lone segment that is contained in the developed representation.

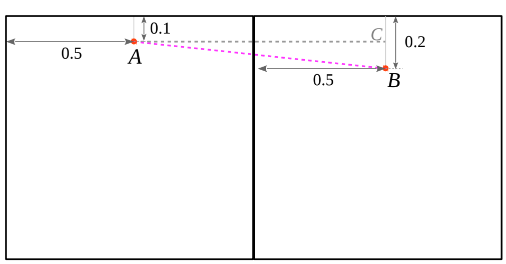
Going back to the example of the unit cube shown in Figure 3(a), one such possible developed representation involves cutting along the edges that are along the Y and Z axes, and preserving the edge along the X axis. This gives the developed representation of Figure 3(b). The potential geodesic distance computed using this developed representation is $\sqrt{(0.5+0.5)^2 + (0.2-0.1)^2} = \sqrt{1.01}$.

However, if we use a different developed representation where we cut along the edge along the X axis, but preserve the edges along the Y and Z axes, as shown in Figure 3(c), the potential geodesic distance computed using this developed representation is $\sqrt{(0.5+0.1)^2 + (0.5+0.2)^2} = \sqrt{0.85}$.

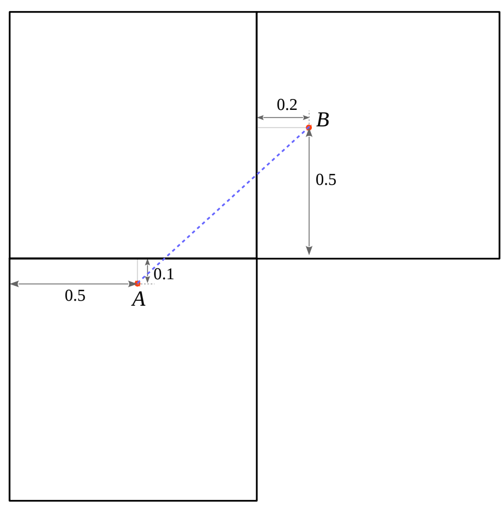
Since the geodesic distance is the length of the shortest path, we need to choose the smaller of the two lengths obtained using the two different developed representations. Thus, $d_G(A,B) = \sqrt{0.85}$.



(a) Points A and B on the cube, and a chosen coordinate system.



(b) The first developed representation obtained by cutting along Y and Z axes (only faces with the points A and B shown).



(c) The second developed representation obtained by cutting along the X axis.

Figure 3: Euclidean and Geodesic distances between two points on the surface of a cube.