Supplementary Notes on Symmetry

Math 170: Ideas in Mathematics (Section 001)

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Notations Illustrated Using the Example of an Equilateral Triangle

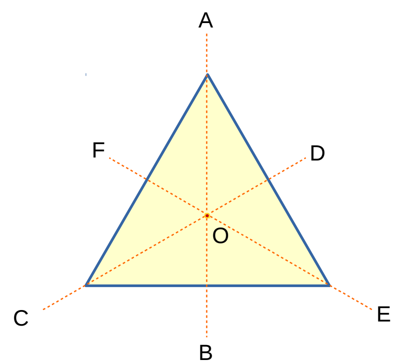

Figure 1: An Equilateral Triangle, T

Let's consider the set of points on the plane that make up the Equilateral triangle, T, as shown in Figure 1. This set remains *invariant* (*i.e.*, remains unchanged) under a reflection about the line $\overline{AB}$. The phrase "reflection about the line $\overline{AB}$" will be concisely written as $\text{Ref}_{\overline{AB}}$, which we will consider as an operation/transformation acting on the set T. Since T remains invariant under $\text{Ref}_{\overline{AB}}$, we write $\text{Ref}_{\overline{AB}}(T) = T$, and call $\text{Ref}_{\overline{AB}}$ a symmetry of T. In particular, the refection symmetries of T are $\text{Ref}_{\overline{AB}}$, $\text{Ref}_{\overline{CD}}$ and $\text{Ref}_{\overline{EF}}$.

Likewise, T has rotational symmetries about O. The simplest of such a symmetry is a rotation of $\frac{2\pi}{3}$ or $120°$. The positive value of the angle implies that it is an anti-clockwise rotation. The entire phrase "rotation about O by an angle of $\frac{2\pi}{3}$" is abbreviated as $\text{Rot}_{O,\frac{2\pi}{3}}$. Thus, $\text{Rot}_{O,\frac{2\pi}{3}}$ is a symmetry of T since $\text{Rot}_{O,\frac{2\pi}{3}}(T) = T$.

Some symmetries are the same: Rotating the triangle by $-\frac{2\pi}{3}$ (*i.e.*. a clockwise rotation of $\frac{2\pi}{3}$) is the same thing as rotating it by $\frac{4\pi}{3}$. Thus we write $\text{Rot}_{O,\frac{4\pi}{3}} = \text{Rot}_{O,\frac{-2\pi}{3}}$. The method for checking this is to track the labels on the vertices upon applying the rotations. If the labels are the same, then they are the same rotations. This is also illustrated in the example of Figure 2.

Composition of Symmetries

We can compose symmetries to obtain more symmetries. Some of these composed symmetries will be new ones, while others may be equivalent to symmetries we already knew of. For example, two consecutive rotations by $\frac{2\pi}{3}$ about O is equivalent to a rotation by $\frac{4\pi}{3}$ about O. That is, $\text{Rot}_{O,\frac{2\pi}{3}} \circ \text{Rot}_{O,\frac{2\pi}{3}} = \text{Rot}_{O,\frac{4\pi}{3}} = \text{Rot}_{O,\frac{-2\pi}{3}}$.

One can compose different symmetries to test if the composition of the symmetries is equivalent to some simpler/more familiar symmetry. The way to determine that, in case of the triangle, for example, is to label the vertices and see what the configuration of the vertices is after the sequence of symmetries are applied. If that final configuration is same for two symmetries, then the symmetries have to be equivalent. An example is shown in Figure 2, which illustrates that (using the shown labels in the figure) $\text{Ref}_N = \text{Rot}_{P,\frac{2\pi}{3}} \circ \text{Ref}_L$ (note that in a composite symmetry joined by "$\circ$", we read/apply the symmetries from the right to left). **(Refer to class notes for more details and examples.)**

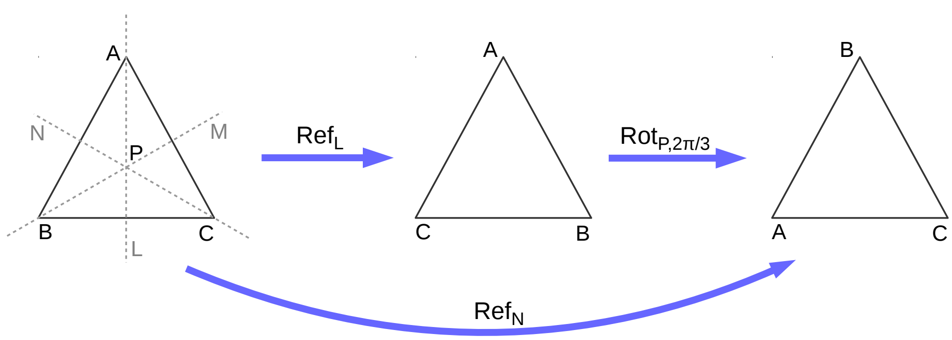

Figure 2: The composite symmetry $\text{Rot}_{P,\frac{2\pi}{3}} \circ \text{Ref}_L$ is same as the symmetry Ref_N

Identity Symmetry, Inverse of a Symmetry and Generating Set

The identity operation/transformation, written as Id, is equivalent to performing no operation/transformation at all! Thus, for example, (once again, using the labeling of Figure 2), $\text{Rot}_{P,2\pi} = \text{Ref}_L \circ \text{Ref}_L = Id$.

For every symmetry S, there exists an *inverse* symmetry, S', such that $S \circ S' = S' \circ S = \text{Id}$. For example, $\text{Rot}_{O,-\frac{2\pi}{3}}$, which is equivalent to $\text{Rot}_{O,\frac{4\pi}{3}}$, is the inverse of $\text{Rot}_{O,\frac{2\pi}{3}}$. A reflection about a line L is the inverse of itself.

On can consider the set, G, of all symmetries of a figure. For example, in case of the equilateral triangle with labels shown in Figure 2, that set is $G = \{\text{Ref}_L,\ \text{Ref}_M,\ \text{Ref}_N,\ \ \text{Rot}_{P,\frac{2\pi}{3}},\ \ \text{Rot}_{P,\frac{4\pi}{3}} = \text{Rot}_{P,-\frac{2\pi}{3}},\ \ \text{Rot}_{P,2\pi} = \text{Rot}_{P,0} = Id\}$, containing 6 elements – the only 6 distinct symmetries of the equilateral triangle.

However, out of the set of all symmetries, one can choose a subset $H \subseteq G$, such that every symmetry in G can be written as a composition of some of the symmetries in H, but none of the symmetries in H can be written as composition of the other symmetries in H. Such a choice of H is called an *independent* or *generating* set of symmetries. For example, in case of the equilateral triangle, a valid choice of a generating set is $H = \{\text{Ref}_L,\ \text{Rot}_{P,\frac{2\pi}{3}}\} \subseteq G$, containing 2 symmetries. It can be easily seen that the remaining 4 symmetries in G can be written as a composition of these two (for example, $\text{Ref}_N = \text{Rot}_{P,\frac{2\pi}{3}} \circ \text{Ref}_L$). **(Refer to class notes for more details and examples.)**

Translational Symmetries and Tiling of the Plane

Figures that have translational symmetries are the tiling of the plane. For example, the square tiling in Figure 3 has symmetries $\text{Trans}_{L,1}$ and $\text{Trans}_{M,1}$, where L and M are directions as shown, and the length of each side of the squares is assumed to be 1 unit. Note that the translation symmetry along two parallel lines are equivalent. Thus, $\text{Trans}_{L,1} = \text{Trans}_{U,1}$ and $\text{Trans}_{M,1} = \text{Trans}_{V,1}$,

The tiling also has reflection symmetries $\text{Ref}_L, \text{Ref}_M$ and reflection symmetry about any of the vertical or horizontal lines that constitute edges of the squares (like L or M) or pass through the centers of the squares (like the lines U and V). It also has rotation symmetries such as $\text{Rot}_{P,\frac{\pi}{4}}$ and $\text{Rot}_{Q,\frac{\pi}{2}}$. In fact the tiling has rotation symmetries about any vertex of the squares or their centers by angles that are multiples of $\frac{\pi}{2}$. Thus there are infinitely many symmetries of a tiling of the plane.

However, not all the symmetries are independent/generating. In fact for a tiling as there, a generating/independent set of symmetries can be quite small. As a rule of thumb, for a tiling, a generating/independent set of symmetries can have *at most* two translation symmetries, one rotation symmetry and one reflection symmetry. For example, for the square tiling shown in Figure 3, the following holds: $\text{Rot}_{Q,\frac{\pi}{2}} = \text{Trans}_{L,1} \circ \text{Trans}_{M,1} \circ \text{Rot}_{P,\frac{\pi}{2}}$ (this can be seen by checking that any point, say A, goes to the same point, $\overline{A}$, upon application of $\text{Rot}_{Q,\frac{\pi}{2}}$ or $\text{Trans}_{L,1} \circ \text{Trans}_{M,1} \circ \text{Rot}_{P,\frac{\pi}{2}}$. This can be checked for any other point as well, such as B going to $\overline{B}$).

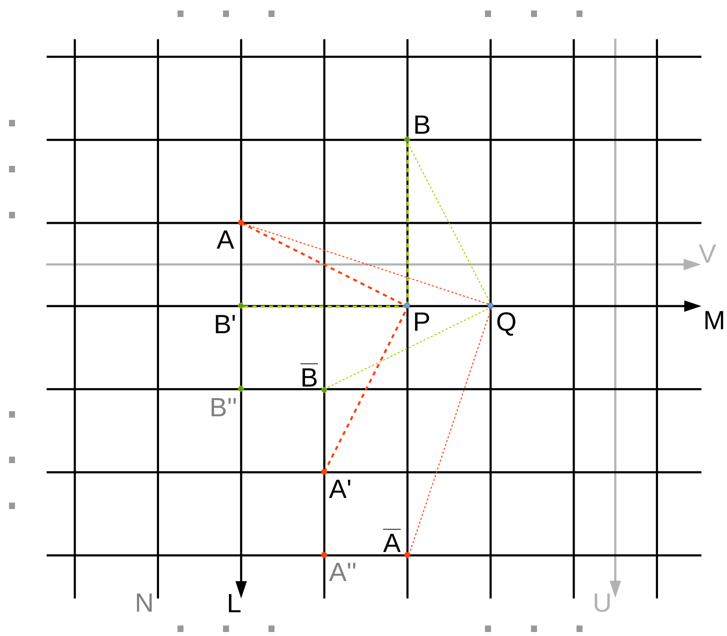

Figure 3: Uniform square tiling of the plane.

For the square tiling a generating/independent set of symmetries can be chosen to be $\{\text{Trans}_{L,1},\ \ \text{Trans}_{M,1},\ \ \text{Rot}_{P,\frac{\pi}{2}},\ \ \text{Ref}_L\}$. Every other symmetry of the square tile can be constructed as a composition of these four. For example,

i. $\text{Rot}_{Q,\frac{\pi}{2}} = \text{Trans}_{L,1} \circ \text{Trans}_{M,1} \circ \text{Rot}_{P,\frac{\pi}{2}}$

ii. $\text{Ref}_N = \text{Trans}_{M,-2} \circ \text{Ref}_L$ (verify this by tracking where the points A and B go when applying each of the transformation on the two sides of the '$=$' sign, just as we did for the above).

iii. etc.

(Refer to class notes for more details and examples.)

Scale Invariance and Fractals

(Refer to class notes for more details and examples.)