Supplementary Notes on Elementary Topology

Math 170: Ideas in Mathematics

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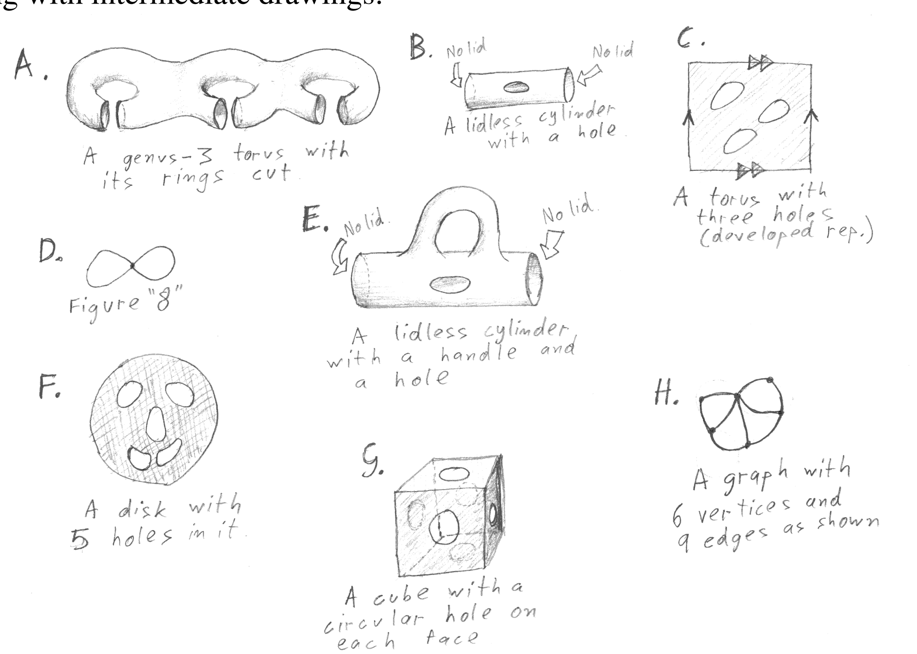
Topologically Equivalent Spaces

Two spaces are called topologically equivalent if one can be changed into the other by bending, pushing, stretching and smoothening operations, but NOT having to use cutting or gluing operations.

Examples: The surface of a dough-nut (a torus) is topologically equivalent to the surface of a coffee mug. The torus with a puncture on it is topologically equivalent to the figure "8" (refer to class notes).

Practice problems:

Classify the spaces A-H drawn in the following figure into topologically equivalent groups. (*i.e.*, write which of these topological spaces are topologically equivalent to which others in the list). Give explanations for your answers along with intermediate drawings:



Euclidean spaces

The 1-dimensional Euclidean space is $\mathbb{R}$. The "picture" for this space is a line that extends to infinity in both directions. Elements/points in this space/set are the usual real numbers, $x \in \mathbb{R}$.

Recall the definition of *Cartesian product*: If A and B are sets, then $A \times B = \{(a,b) \mid a \in A, b \in B\}$.

The 2-dimensional Euclidean space is the *Cartesian product* of two copies of $\mathbb{R}$. It is thus $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\}$. The "picture" for this space is a plane that extends to infinity in all the directions along the plane. Elements/points in this space/set are pairs of real numbers, $(x,y) \in \mathbb{R}^2$.

The 3-dimensional Euclidean space is the *Cartesian product* of three copies of $\mathbb{R}$. It is thus $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$. The "picture" for this space is the usual 3-dimensional space that we live in, which extends to infinity in all the directions. Elements/points in this space/set are 3-tuple of real numbers, $(x,y,z) \in \mathbb{R}^3$.

The 4-dimensional Euclidean space is the *Cartesian product* of four copies of $\mathbb{R}$. It is thus $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z,w) \mid x,y,z,w \in \mathbb{R}\}$. This does not have a nice "picture" that we can refer to. But it is absolutely well-defined as a Cartesian product of 4 copies of $\mathbb{R}$. Elements/points in this space/set are 4-tuple of real numbers, $(x,y,z,w) \in \mathbb{R}^4$.

We can continue this process and define n-dimensional Euclidean space in general, which is the *Cartesian product* of n copies of $\mathbb{R}$.

Manifolds and their Dimensions

A topological space is called a *manifold* if locally at ***every point*** it looks like a Euclidean space. If locally it looks like $\mathbb{R}$ (a line), it is called a 1-dimensional manifold, if locally it looks like $\mathbb{R}^2$ (a plane), it is called a 2-dimensional manifold, if locally it looks like $\mathbb{R}^3$ (the 3-dimensional Euclidean space), it is called a 3-dimensional manifold, and so on.
Examples:

The 1-dimensional Euclidean space, $\mathbb{R}$, is a 1-dimensional manifold. A circle, $\mathbb{S}^1$, is also a 1-dimensional manifold.

The figure "8" is not a manifold since there is a point (at the crossing of the curve) where it locally looks like a "$\times$" shape.

The 2-dimensional Euclidean space and the usual 2-dimensional sphere (the surface of a ball), $\mathbb{S}^2$, are 2-dimensional manifolds. The usual torus (surface of a doughnut), the genus-2 torus, the genus-3 torus, etc. – all are 2-dimensional manifolds.

A 2-dimensional sphere, $\mathbb{S}^2$, attached to a loop/circle at a point on it is not a manifold.

The 3-dimensional Euclidean space is a 3-dimensional manifold. So is the 3-dimensional sphere, $\mathbb{S}^3$.

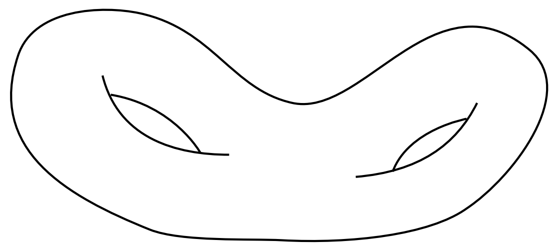
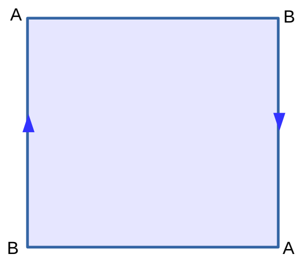


Figure 1: Genus-2 torus.

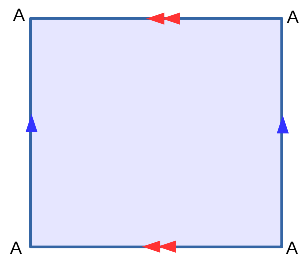
Cutting and Gluing

(Refer to Class Lectures and the book)

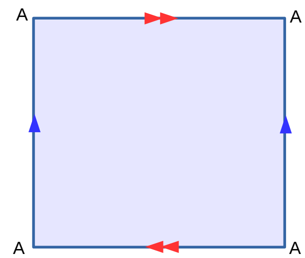
Example: Gluing along the same-colored arrows in the Figure 2 (and matching their



(a) Developed representation of the Mobius Band



(b) Developed representation of the Torus



(c) Developed representation of the Klein bottle

Figure 2: Developed Representations.

directions) gives us the respective topological spaces.

Gluing the entire boundary of a 2-dimensional disk to a single point gives us the 2-dimensional sphere, $\mathbb{S}^2$.

First Betti Number of a 2-dimensional manifold

The first Betti number of a 2-dimensional manifold is the maximum number of closed curves that you can draw on the manifold such that cutting along those closed curves will not disconnect the space (*i.e.*, will not create more than one pieces).

Examples:

The first Betti number of the usual torus is 2. The first Betti number of the sphere, $\mathbb{S}^2$, is 0 (zero).

Simply-connected Space

A topological space is called simply-connected if any closed loop in the space can be contracted to a point in a continuous manner, while keeping the loop in the space all throughout. Examples of simply-connected spaces: $\mathbb{R}^2, \mathbb{S}^2, \mathbb{S}^3$. Examples of spaces that are not simply-connected: $\mathbb{S}^1$, the usual torus, torus of genus 2 and higher.

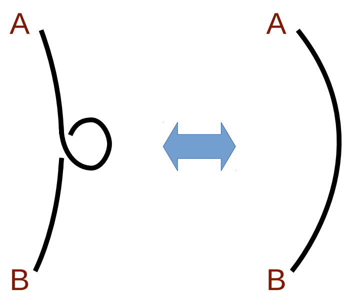
Knot Theory

Knots are embeddings of the circle in $\mathbb{R}^3$. Knot diagrams are projections of the knots on a plane. Examples of knots are shown in Figure 4.

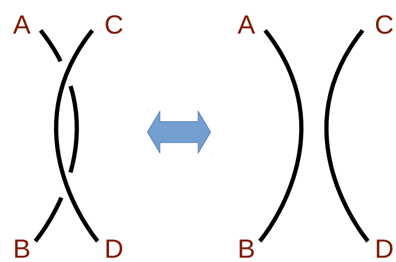
(Refer to Class Lectures for more details)

Reidemeister moves

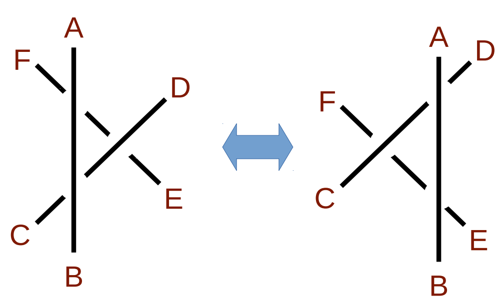
Reidemeister moves are "moves" made involving crossings in a knot upon performing which the knot remains equivalent to the original one. There are three Reidemeister moves. It has been shown that if two knot diagrams are equivalent as knots, then the three types of Reidemeister moves are sufficient to go from one diagram to another. The three Reidemeister moves are shown below:



(a) Reidemeister move of type 1



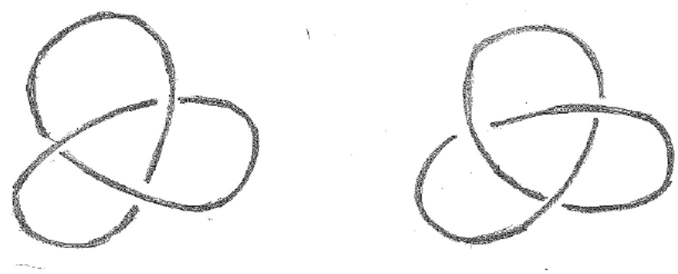
(b) Reidemeister move of type 2



(c) Reidemeister move of type 3

Figure 3: Reidemeister moves

Two knots are *knot equivalent* if one can be transformed into the other using only the Reidemeister moves. An example using the Reidemeister moves is shown in Figure 5.



Trefoil knots (right-handed and left-handed)

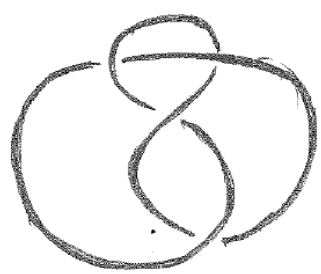
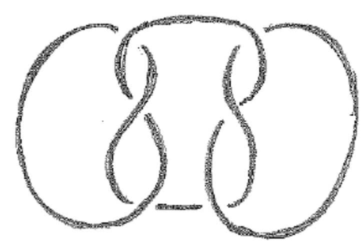
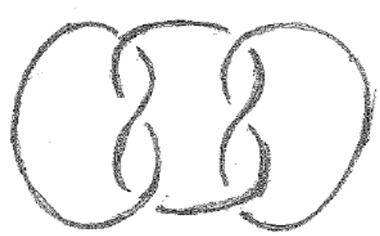


Figure-8 Knot



Square Knot



Granny Knot

Figure 4: Examples of Knots

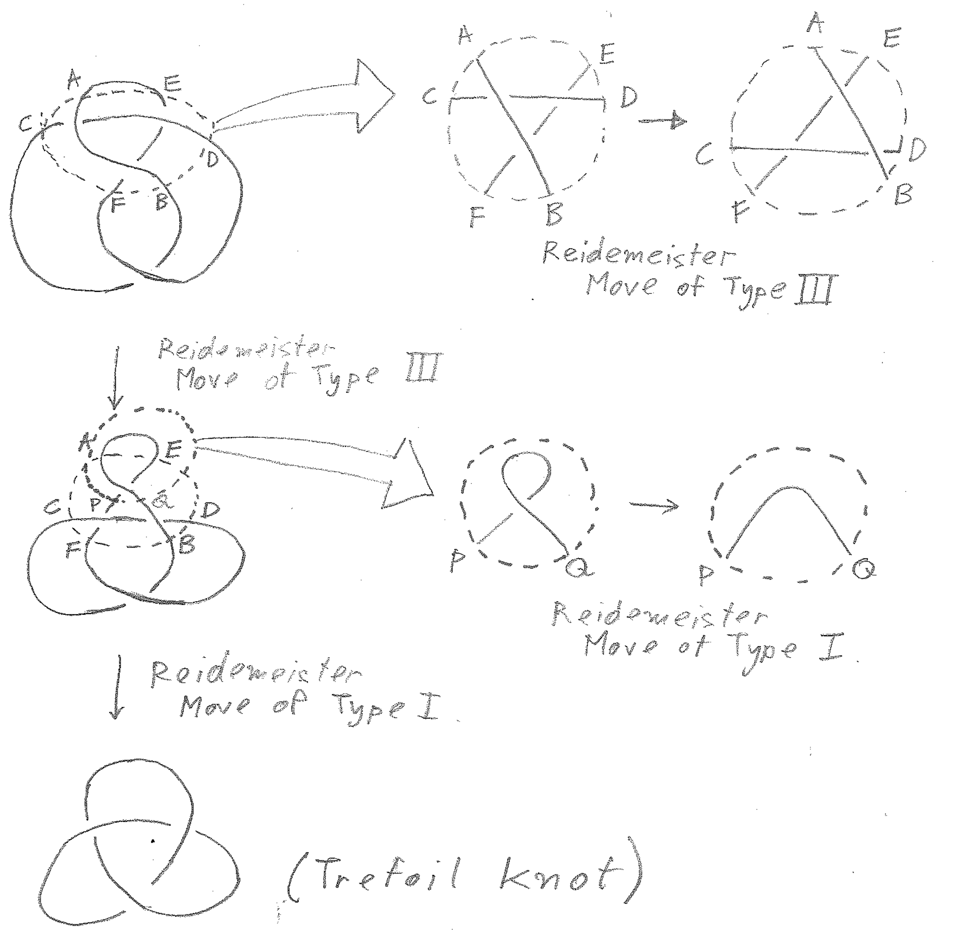


Figure 5: Examples of Knot equivalence illustrated using the Reidemeister moves.