Kalman Filter Basics

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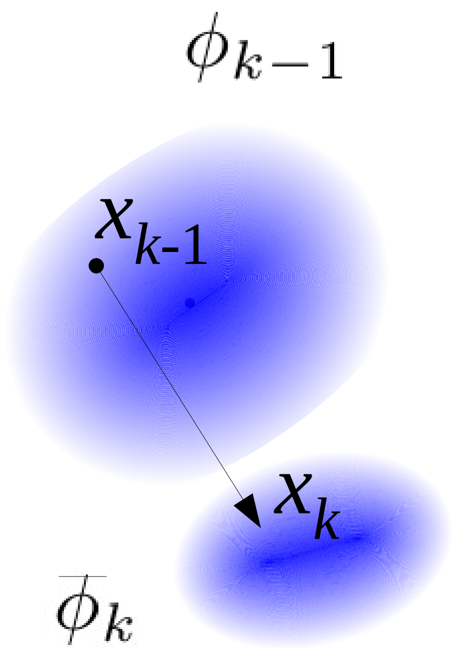
Kalman Filter Basics



System state is described by $x \in \mathbb{R}^N$

Exact state is not known, but described by a Normal probability distribution, $\phi \sim \mathcal{N}(\mu, \Sigma)$

$$\phi(x) \;=\; \frac{1}{\sqrt{(2\pi)^N|\Sigma|}}\exp\Big[-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big], \quad x \in \mathbb{R}^N$$

Deterministic system evolution: $x_k \;=\; F_k x_{k-1} + B_k u_k \;+ w_k$

Probabilistic system evolution:

$$w_k \sim \mathcal{N}(0, Q_k)$$

$$\overline{\phi}_k(x_k) \;=\; \int_{\mathbb{R}^N} \frac{1}{|\nabla f|}\; \phi_{k-1}\circ f^{-1}(x)\;\; w_k(x_k - x)\;\; dx$$

where, $f(x) = F_k x + B_k u_k$

In general, if ϕ_{k-1} is Normal, ϕ_k is not Normal.

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Deterministic system evolution: $x_k \;=\; F_k x_{k-1} + B_k u_k \;+ w_k$

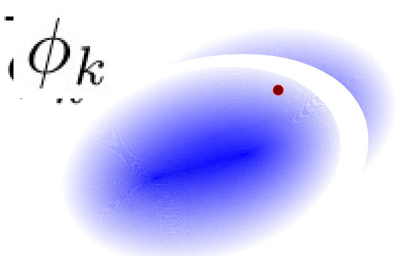
$$w_k \sim \mathcal{N}(0, Q_k)$$

Probabilistic system evolution
(*Prediction* step of Kalman Filter):

$$\begin{aligned}\overline{\mu}_k &= F_k \mu_{k-1} \;+\; B_k u_k \\ \overline{\Sigma}_k &= F_k \Sigma_{k-1} F_k^T \;+\; Q_k\end{aligned}$$

$$\phi_{k-1} \sim \mathcal{N}(\mu_{k-1}, \Sigma_{k-1})$$
$$\overline{\phi}_k \sim \mathcal{N}(\overline{\mu}_k, \overline{\Sigma}_k)$$

Incorporating measurements:



$$z_k \;=\; H_k x_k + v_k$$
$$v_k \sim \mathcal{N}(0, R_k)$$

Update **step of Kalman Filter:**

$$\begin{aligned}\mu_k &= \overline{\mu}_k + K_k(z_k - H\overline{\mu}_k) \\ \Sigma_k &= (I - K_k H_k)\overline{\Sigma}_k\end{aligned}$$

$$\text{where, } K_k = \overline{\Sigma}_k H_k^T (H_k \overline{\Sigma}_k H_k^T + R_k)^{-1}$$

Bayesian inference:

$$P(\text{state is } x_k \,|\, \text{observation is } z_k) \propto P(\text{observation is } z_k \,|\, \text{state is } x_k)\; P(\text{state is } x_k)$$

$$\Rightarrow \quad \phi_k(x_k) \;\propto\; v_k(z_k - H_k x_k)\; \overline{\phi}_k(x_k)$$

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input(s)

System dynamics: $$x_k = F_k x_{k-1} + B_{k-1}u_{k-1} + E_{k-1}d_{k-1} + w_k$$

Motion model: $$z_k = H_k x_k + v_k$$

$$x_k \in \mathbb{R}^n \quad z_k \in \mathbb{R}^m \quad u_k \in \mathbb{R}^p \quad d_k \in \mathbb{R}^q$$

$$w_k \sim \mathcal{N}(0, Q_k)$$
$$v_k \sim \mathcal{N}(0, R_k)$$

Kalman Filter summarized:

$$\mu_k = (I - K_k H_k)\left(F_k \mu_{k-1} + B_{k-1}u_{k-1} + E_{k-1}d_{k-1}\right) + K_k z_k$$
$$\Sigma_k = (I - K_k H_k)\left(F_k \Sigma_{k-1} F_k^T + Q_k\right)$$

Riccati Difference Eqn.

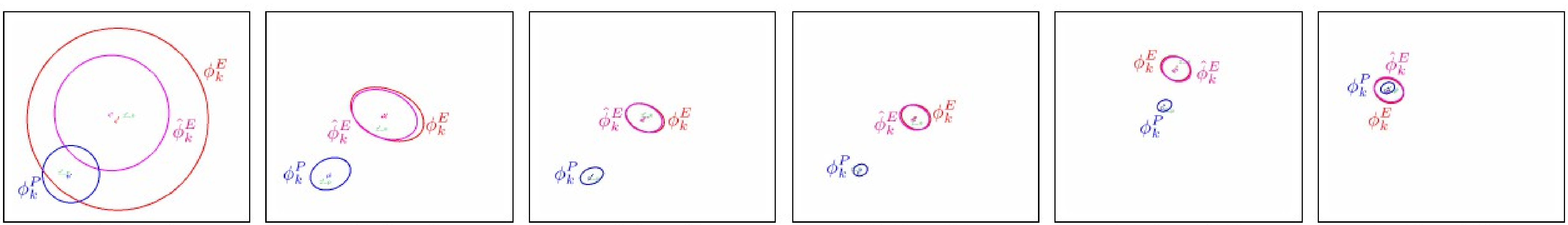
where,
$$K_k = \left(F_k \Sigma_{k-1} F_k^T + Q_k\right) H_k^T \left(H_k \left(F_k \Sigma_{k-1} F_k^T + Q_k\right) H_k^T + R_k\right)^{-1}$$

We will concisely write these recursive relations as:

$$\{\phi_0, \phi_1, \phi_2, \cdots\} = \mathcal{KF}_{\mathcal{P}}(\phi_0; \{[u_0, d_0], [u_1, d_1], \cdots\}; \{z_1, z_2, \cdots\})$$

where, $\mathcal{P} = (\{F_k\}, \{B_k\}, \{E_k\}, \{H_k\}, \{Q_k\}, \{R_k\})$

Example: Pursuit-Evasion Game for Normal Distributions



(a) $k = 1$ (b) $k = 3$ (c) $k = 11$ (d) $k = 50$ (e) $k = 501$ (f) $k = 1001$