Configuration Spaces and Topology †

Supplementary Notes – Part 1.b
Example of a non-Holonomic System

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Recall that the configuration space of a single point mobile robot navigating in a unbounded 2-dimensional flat plane with obstacles is simply $\mathbb{R}^2 - \mathcal{O}$, where $\mathcal{O}$ represents the set of points on the plane that make up the obstacles (Figure 1).

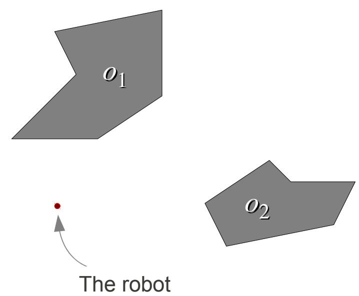


Figure 1: The configuration space of a point robot navigating on a plane with obstacles o_1 and o_2 is $\mathbb{R}^2 - (o_1 \cup o_2)$.

Now, let us consider a *unicycle* model of a point robot [2] navigating in $(\mathbb{R}^2 - \mathcal{O})$. That means, in addition to the position (x, y), the robot has an orientation (θ). Thus the configuration space of the robot is now $\mathcal{C} = (\mathbb{R}^2 - \mathcal{O}) \times \mathbb{S}^1$ (a subset of $SE(2) \equiv \mathbb{R}^2 \times SO(2)$).

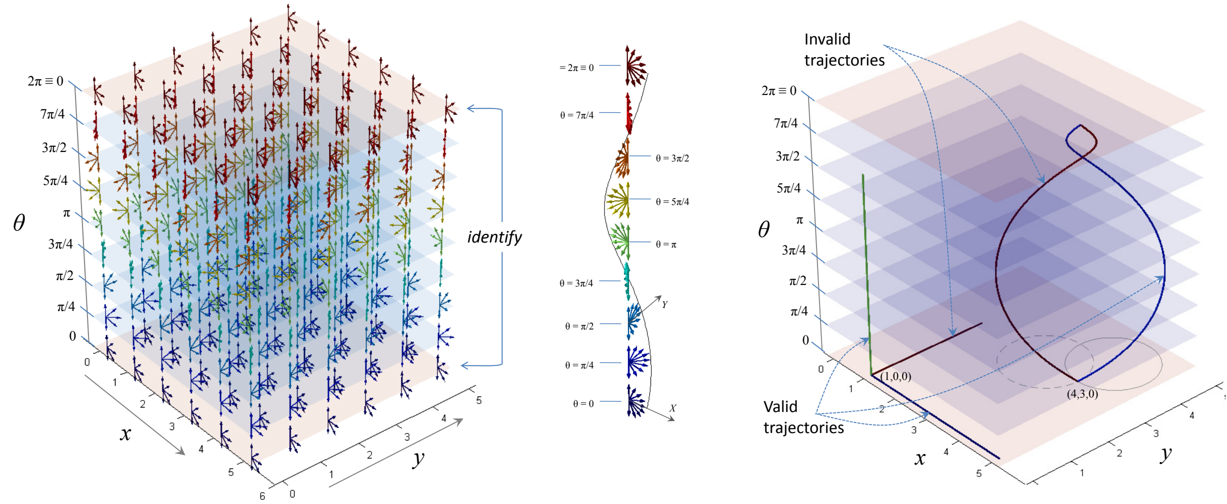
The unicycle model also implies that the robot can only move forward along the direction it is oriented, or rotate at a fixed place (*i.e.* x, y remains fixed, while θ changes). So at point $p = (x, y, \theta)$ in its configuration space, the robot can move along $(\dot{x}, \dot{y}, \dot{\theta}) = (v\cos(\theta), v\sin(\theta), \omega)$, for some $v \in \mathbb{R}_+$ and $\omega \in \mathbb{R}$ (representing forward and angular speeds). Thus the possible directions of motion is a 2-dimensional manifold generated by v and ω. This is the space of possible actions at (x, y, θ), which, in general, may not be a vector space, and in this particular example is a half space (a subset of the tangent space $T_p\mathcal{C}$). Attaching this space of possible actions to every point of the C-space (Figure 2(a)), we obtain a fiber bundle. The fiber bundle itself is a 5-dimensional manifold, sections of which give vector fields in the configuration space. A valid trajectory in the C-space needs to be such that the tangent at every point on it lies in the set of possible actions at that point (Figure 2(c)).

References

[1] Subhrajit Bhattacharya. *Topological and Geometric Techniques in Graph-Search Based Robot Planning.* PhD thesis, University of Pennsylvania, January 2012.

[2] B. d'Andrea Novel, G. Campion, and G. Bastin. Control of nonholonomic wheeled mobile robots by state feedback linearization. *The International Journal of Robotics Research*, 14(6), Sept. 1995.

†Adapted from [1]



(a) The configuration space $\mathbb{R}^2 \times \mathbb{S}^1$, and the action space attached to every point in it (the arrows in the figure are some representative actions – the action spaces themselves are half-spaces).

(b) Closer look at how the space of possible actions change with θ.

(c) Some valid and invalid trajectories in the C-space. The tangent at every point on the trajectory must lie inside the action fiber at the point.

Figure 2: Configuration space of a unicycle point robot, and actions due to kinematic constraints. For simplicity we have eliminated obstacles in the environment.