Path Homotopy Invariants and their Application to Optimal Trajectory Planning

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Abstract

We consider the problem of optimal path planning in different homotopy classes in a given environment. Though important in robotics applications, path-planning with reasoning about homotopy classes of trajectories has typically focused on subsets of the Euclidean plane in the robotics literature. The problem of finding optimal trajectories in different homotopy classes in more general configuration spaces (or even characterizing the homotopy classes of such trajectories) can be difficult. In this paper we propose automated solutions to this problem in several general classes of configuration spaces by constructing presentations of fundamental groups and giving algorithms for solving the *word problem* in such groups. We present explicit results that apply to knot and link complements in 3-space, discuss how to extend to cylindrically-deleted coordination spaces of arbitrary dimension, and also present results in the coordination space of robots navigating on an Euclidean plane.

1 Introduction

In the context of robot motion planning, one often encounters problems requiring optimal trajectories (paths) in different homotopy classes. For example, consideration of homotopy classes is vital in planning trajectories for robot teams separating/caging and transporting objects using a flexible cable [Bhattacharya et al., 2015], or in planning optimal trajectories for robots that are tethered to a base using a fixed-length flexible cable [Kim et al., 2014]. This paper addresses the problem of optimal path planning with homotopy class as the optimization constraint.

There is, certainly, a large literature on minimal path-planning in computational geometry (for a brief sampling and overview, see [Mitchell and Sharir, 2004]) Of course, since the problem of computing shortest paths (even for a 3-d simply-connected polygonal domain) is NP-hard [Canny and Reif, 1987], we must restrict attention to subclasses of spaces, even when using homotopy path constraints. This paper focuses on two interesting and completely different types: (1) knot and link complements in 3-d; and (2) cylindrically-deleted coordination spaces [Ghrist and LaValle, 2006].

2 Configuration Spaces with Free Fundamental Groups

2.1 Motivation: Homotopy Invariant in $(\mathbb{R}^2 - \mathcal{O})$

We are interested in constructing computable *homotopy invariants* for trajectories in a configuration space that are amenable to graph search-based path planning. To that end there is a very simple construction for configuration spaces of the form $\mathbb{R}^2 - \mathcal{O}$ (Euclidean plane punctured by obstacles) [Grigoriev and Slissenko, 1998, Hershberger and Snoeyink, 1991, Tovar et al., 2008, Bhattacharya et al., 2015, Kim et al., 2014]: We start by placing *representative points*, ζ_i, inside the ith connected component of the obstacles, $O_i \subset \mathcal{O}$. We

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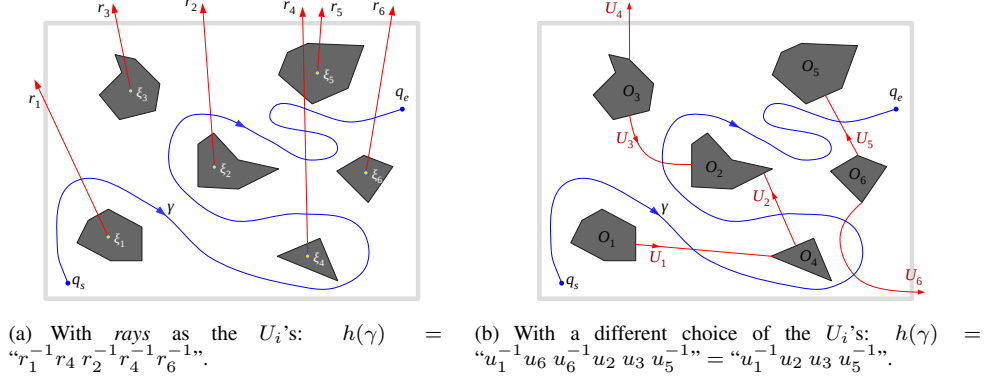[†]Department of Mathematics, University of Pennsylvania, PA.

(a) With *rays* as the U_i's: $h(\gamma) = $ "$r_1^{-1} r_4 \, r_2^{-1} r_4^{-1} r_6^{-1}$".

(b) With a different choice of the U_i's: $h(\gamma) = $ "$u_1^{-1} u_6 \, u_6^{-1} u_2 \, u_3 \, u_5^{-1}$" = "$u_1^{-1} u_2 \, u_3 \, u_5^{-1}$".

Figure 1: Homotopy invariants of curves such as γ in $(\mathbb{R}^2 - \mathcal{O})$ are words constructed by tracing γ and inserting letters in the word for every crossing of the chosen oriented sub-manifolds, U_i (in red).

then construct non-intersecting rays, $r_1, r_2, \cdots, r_m$, emanating from the representative points (this is always possible, for example, by choosing the rays to be parallel to each other). Now, given a curve γ in $\mathbb{R}^2 - \mathcal{O}$, we construct a *word* by tracing the curve, and every time we cross a ray r_i from its right to left, we insert the letter "r_i" into the word, and every time we cross it from left to right, we insert a letter "r_i^{-1}" into the word, with consecutive r_j and r_j^{-1} canceling each other. The word thus constructed is written as $h(\gamma)$. For example, in Figure 1(a), $h(\gamma) = $ "$r_1^{-1} r_4 \, r_2^{-1} r_4^{-1} r_4 \, r_4^{-1} r_6^{-1}$" = "$r_1^{-1} r_4 \, r_2^{-1} r_4^{-1} r_6^{-1}$". This word, called the *reduced word* for the trajectory γ, is a complete homotopy invariant for trajectories connecting the same set of points. That is, $\gamma_1, \gamma_2 : [0,1] \to (\mathbb{R}^2 - \mathcal{O})$, with $\gamma_i(0) = q_s, \gamma_i(1) = q_e$ are homotopic if and only if $h(\gamma_1) = h(\gamma_2)$.

2.2 Words as Homotopy Invariants in Spaces with Free Fundamental Groups

In a more general setting the aforesaid construction can be generalized as follows:

Construction 1 *Given a D-dimensional manifold (possibly with boundary), X, suppose $U_1, U_2, \cdots U_n$ are $(D-1)$-dimensional orientable sub-manifolds (not necessarily smooth and possibly with boundaries) such that $\partial U_i \subseteq \partial X$. Then, for any curve, γ (connecting fixed start and end points, $x_s, x_e \in X$), which is in general position (transverse) w.r.t. the U_i's, one can construct a word by tracing the curve and inserting into the word a letter, u_i or u_i^{-1}, whenever the curve intersects U_i with a positive or negative orientation respectively.*

The proposition below is a direct consequence of a simple version of the Van Kampen's Theorem (of which several different generalizations are available in the literature).

Proposition 1 *Words constructed as described in Construction 1 are complete homotopy invariants for curves in X joining the given start and end points if the following conditions hold:*

1. $U_i \cap U_j = \emptyset, \ \forall i \neq j$.

2. $X - \bigcup_{i=1}^{n} U_i$ is path connected and simply-connected, and,

3. $\pi_1(X - \bigcup_{i=1, i \neq j}^{n} U_i) \simeq \mathbb{Z}, \ \forall j = 1, 2, \cdots, n$,

Proof. Consider the spaces $X_0 = X - \bigcup_{i=1}^{n} U_i$ and $X_j = X - \bigcup_{i=1, i \neq j}^{n} U_i, \ j = 1, 2, \cdots, n$. Due to the aforesaid properties of the U_i's the set $\mathcal{C}_X = \{X_0, X_1, \cdots, X_n\}$ constitutes an open cover of X, is closed under intersection, the pairwise intersections $X_i \cap X_j = X_0, i \neq j$ are simply-connected (and hence path connected), and so are $X_i \cap X_j \cap X_k = X_0, i \neq j \neq k$.
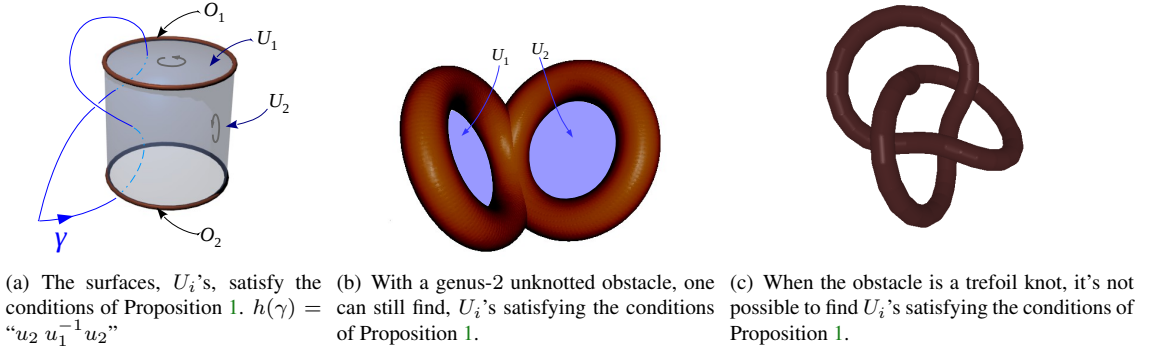
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(a) The surfaces, U_i's, satisfy the conditions of Proposition 1. $h(\gamma) =$ "$u_2\,u_1^{-1}u_2$"

(b) With a genus-2 unknotted obstacle, one can still find, U_i's satisfying the conditions of Proposition 1.

(c) When the obstacle is a trefoil knot, it's not possible to find U_i's satisfying the conditions of Proposition 1.

Figure 2: The fundamental groups of configuration spaces, $\mathbb{R}^3 - \mathcal{O}$, may or may not be free.

The proof, when γ is a closed loop (*i.e.* $x_s = x_e$), then follows directly from the Seifert-van Kampen theorem [Hatcher, 2001, Crowell, 1959] by observing that $\pi_1(X) \simeq \pi_1(X_0) * \pi_1(X_1) * \pi_1(X_2) * \cdots * \pi_1(X_n) \simeq *_{i=1}^n \mathbb{Z}$, the free product of n copies of $\mathbb{Z}$, each $\mathbb{Z}$ generated due to the restriction of the curve to X_i, $i = 1, 2, \cdots, n$.

When γ_1 and γ_2 are curves (not necessarily closed) joining points x_s and x_e, they are in the same homotopy class iff $\gamma_1 \cup -\gamma_2$ is null-homotopic — that is, $h(\gamma_1) \circ h(-\gamma_2) = \text{“ ”} \Leftrightarrow h(\gamma_1) = h(\gamma_2)$ (where by "$\circ$" we mean word concatenation).

$\square$

The Construction 1 gives a *presentation* [Epstein, 1992] of the fundamental group of X (which, in this case, is a free group due to the Van Kampen's theorem) as the group generated by the set of letters $\mathbf{U} = \{u_1, u_2, \cdots, u_q\}$, and is written as $G = \pi_1(X) = < u_1, u_2, \cdots, u_q > = < \mathbf{U} >$. In our earlier construction with the rays, $X = \mathbb{R}^2 - \mathcal{O}$ was the configuration space, and $U_i = X \cap r_i$ were the support of the rays in the configuration space. It is easy to check that the conditions in the above proposition are satisfied with these choices. However such choices of rays is not the only possible construction of the U_i's satisfying the conditions of Proposition 1. Figure 1(b) shows a different choice of the U_i's that satisfy all the conditions.

2.3   Simple Extension to $(\mathbb{R}^3 - \mathcal{O})$ with Unlinked Unknotted Obstacles

The construction described in Section 2.1 can be easily extended to the 3-dimensional Euclidean space punctured by a finite number of un-knotted and un-linked toroidal (possibly of multi-genus) obstacles. Instead of "rays", in this case the U_i's are 2-dimensional sub-manifolds that satisfy the conditions in Proposition 1, with a letter, u_i (or u_i^{-1}), being inserted in $h(\gamma)$ every time the curve, γ, crosses/intersects a U_i. This is illustrated in Figures 2(a) and 2(b).

However, a little investigation makes it obvious that such 2-dimensional sub-manifolds cannot always be constructed when the obstacle are knotted or linked (Figure 2(c)). One can indeed construct surfaces (*e.g.* Seifert surfaces) satisfying some of the properties, but not all.

2.4   Application to Graph Search-based Path Planning

Using the homotopy invariants described in the previous sub-section, we describe a graph construction for use in search-based path planning for computing optimal (in the graph) trajectories in different homotopy classes. We first fix the set of sub-manifolds $\{U_1, U_2, \cdots, U_n\}$ as described earlier. Now, given a discrete graph representation of the configuration space, $G = (V, E)$ (*i.e.*, the vertex set, V, consists of points in X, and the edge set, E, contains edges that connect *neighboring* vertices) such that $x_s \in V$, we construct an h-augmented graph, $G_h = (V_h, E_h)$, which is essentially a *lift* of G into the *universal covering space* of X [Hatcher, 2001]. The construction of such augmented graphs has been described in our prior work [Bhattacharya et al., 2015, Kim et al., 2014], and the explicit construction of G_h can be described as follows:
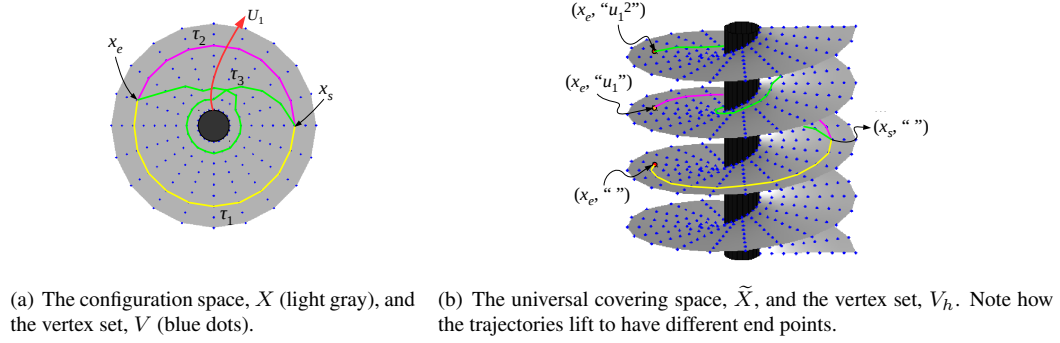
(a) The configuration space, X (light gray), and the vertex set, V (blue dots).

(b) The universal covering space, $\widetilde{X}$, and the vertex set, V_h. Note how the trajectories lift to have different end points.

Figure 3: The h-augmented graph, G_h, is a lift of G into $\widetilde{X}$.

i. Vertices in V_h are tuples of the form (x, w), where $x \in V$ and w is a word made out of letters u_i and u_i^{-1}.

ii. $(x_s, \text{``''}) \in V_h$.

iii. For every edge $[x_1, x_2] \in E$ and every vertex $(x_1, w) \in V_h$, there exists an edge $(x_2, w \circ h(\overrightarrow{x_1 x_2})) \in E_h$, where $\overrightarrow{x_1 x_2}$ denotes the directed curve that constitutes the edge $[x_1, x_2]$.

iv. The length/cost of an edge in G_h is same as its projection in G: $C_{G_h}([(x_1, w_1), (x_2, w_2)]) = C_G([x_1, x_2])$.

The item 'i.' is just a qualitative description of the vertices in G_h. Item 'ii.' describes one particular vertex in G_h, and using that, item 'iii.' describes an incremental construction of the entire graph G_h. The topology of G_h can be described as a lift of G into the universal covering space, $\widetilde{X}$, of X, and is illustrated in Figure 3 for a uniform cylindrically discretized space with a single disk-shaped obstacle.

Such an incremental construction is well-suited for use in graph search algorithms such as Dijkstra's or A* [Cormen et al., 2001], in which one initiates an *open set* using the start vertex (in item ii.), and then gradually *expands* vertices, generating only the neighbors at every expansion (the recipe for which is given by item 'iii.'). Executing a search (Dijkstra's/A*) in G_h from $(x_s, \text{``''})$ to vertices of the form $(x_e, *)$ (where '$*$' denotes any word), and projecting it back to G, gives us optimal trajectories in G that belong to different homotopy classes. Figure 4(a) shows 5 such optimal trajectories in the graph, connecting a given start and goal vertex, where G was constructed by an uniform hexagonal discretization of the planar configuration space. One can then employ a simple curve shortening algorithm [Kim et al., 2014] to obtain ones more optimal than the ones restricted to G (Figure 4(b)). Similarly, shortest trajectories connecting x_s and x_e can be obtained in 3-dimensional configuration spaces (Figure 4(d)) with free fundamental group.

3 Knot and Link Complements

As described earlier, when the obstacle set in $\mathbb{R}^3$ consists of knots and links, it is in general not possible to find the sub-manifolds $U_i \subset (\mathbb{R}^3 - \mathcal{O})$ as required by Proposition 1. However, thankfully we have more generalized versions of the Van Kampen theorem at our disposal that lets us extend the proposed methodology to such spaces. We first illustrate the generalization in $\mathbb{R}^3 - \mathcal{O}$ using knot/link diagrams.

3.1 Dehn Presentation of Fundamental Group of Knot/Link Complements

For simplicity we consider knots and links in $\mathbb{R}^3$ as obstacles. We assume that the knots/links are described by polygons, all of which together constitute $O \subset \mathbb{R}^3$. The *thickened* obstacles (the knots/links with the tubular neighborhoods) will be referred to as $\mathcal{O}$. We consider a knot/link diagram [Lickorish, 1997] of the obstacles: Given a projection map, $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, the knot/link diagram is the projection of the knot/link, $p(O)$, along with additional information about the *z-ordering* at the self-intersections of $p(O)$. We assume that

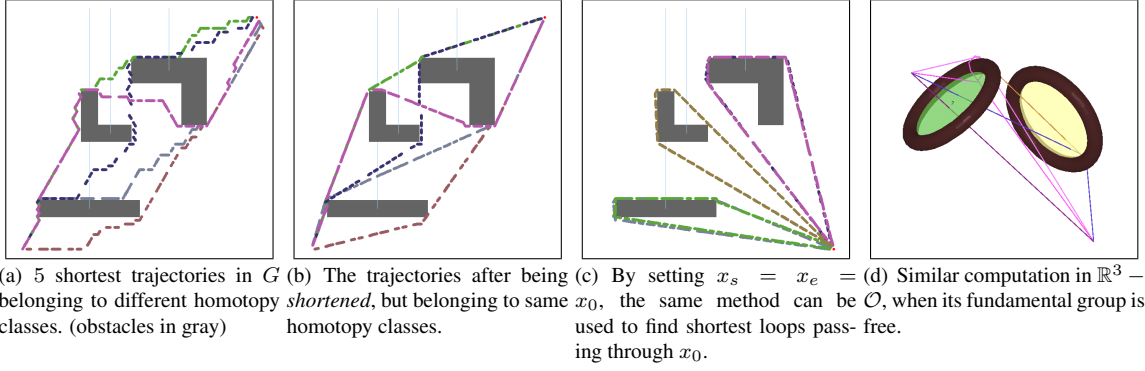(a) 5 shortest trajectories in G belonging to different homotopy classes. (obstacles in gray)

(b) The trajectories after being *shortened*, but belonging to same homotopy classes.

(c) By setting $x_s = x_e = x_0$, the same method can be used to find shortest loops passing through x_0.

(d) Similar computation in $\mathbb{R}^3 - \mathcal{O}$, when its fundamental group is free.

Figure 4: Simple results in configuration spaces that have free fundamental groups. The dot/dash pattern and colors are shown to distinguishing between the trajectories.

(a) Knot diagram, showing one of the polygons, Q_3 (in cyan).

(b) The surfaces, U_i, shown in different colors. The closed loop γ is null-homotopuc, but $h(\gamma) = ``u_1^{-1} u_2 u_3^{-1}"$. Top and side views.
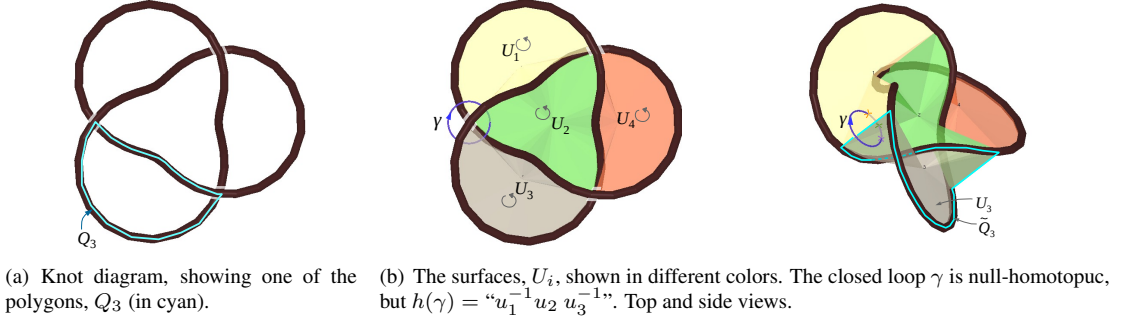
Figure 5: Constructing the surfaces, U_i, from polygonal knot/link diagrams (polygon segments shown as thickened cylinders for easy visualization). Null-homotopic loops as γ have non-empty words.

in this diagram the self-intersections are all transverse (which can always be achieved through infinitesimal perturbations) and that the diagram divides the plane into simply-connected regions (say q counts of them) each bounded by segments of the projected obstacles, and one unbounded exterior region. The *boundary* (the boundary of the closure) of each of the bounded regions is itself a polygon, $Q_i \subseteq p(O)$, $i = 1, 2, \cdots, q$ (Figure 5(a)). Clearly $p^{-1}(Q_i) \cap O$ (the preimage of Q_i in the original obstacle) will be a discontinuous polygon, with discontinuities at the preimages of the self-intersection points on the knot diagram. But these discontinuities can be removed simply by "connecting" the preimages at each self-intersection point, resulting into a spatial polygon, $\widetilde{Q}_i$ with the property that $p(\widetilde{Q}_i) = Q_i$. A simple triangulation can then be employed to construct a surface, U_i, in $\mathbb{R}^3 - O$, such that its boundary is $\widetilde{Q}_i$ and $p(U_i)$ is the simply-connected region bounded by Q_i (this can be achieved by first triangulating the planar region, $p(U_i)$, and then *lifting* the triangulation to $\mathbb{R}^3$) – see Figure 5(a).

The U_i's thus constructed satisfy properties (2) and (3) of Proposition 1, but not property (1), nor do they satisfy the property $\partial U_i \subseteq \partial X$. The consequence of this is that near the regions where the U_i's intersect, there can be closed loops in $\mathbb{R}^3 - O$ which are null-homotopic, but words constructed simply by tracing the loop and inserting letters corresponding to intersections with the U_i's, as we did earlier, may not be the empty word (identity element). An example is illustrated in Figure 5(b)). Due to our construction, such intersection of the U_i's happen only along lines passing through the pre-image of the self-intersections in the knot diagram, for each of which we end up getting a null-homotopic closed loop with non-empty word.

The Dehn presentation [Weinbaum, 1971] uses surfaces as constructed to describe the fundamental group of knot/link complements. We consider the free group, $G = < u_1, u_2, \cdots, u_q > = < \mathbf{U} >$. In general, for every self-intersection in the knot/link diagram, there are four adjacent surfaces, $U_{i_1}, U_{i_2}, U_{i_3}$ and U_{i_4}
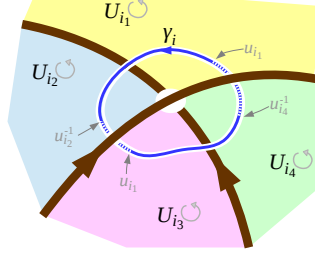
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Figure 6: A "self-intersection" in a knot/link diagram, with a null-homotopic loop, γ_i, intersecting the surfaces adjacent to the intersection. $h(\gamma_i) = "u_{i_1} u_{i_2}^{-1} u_{i_3} u_{i_4}^{-1}"$. One or more of the surfaces can be non-existent, in which case the corresponding letters are simply absent from the word. These words constitute $\mathbf{R}$, and should map to the identity element in $\pi_1(X) = <\mathbf{U} \mid \mathbf{R}>$

in the order as shown in Figure 6 (when the self-intersection is adjacent to the unbounded region in the knot diagram, there are only three). Correspondingly, there is a closed null-homotopic loop, γ_i, that has a word $\rho_i = "u_{i_1} u_{i_2}^{-1} u_{i_3} u_{i_4}^{-1}"$. Thus we have such words $\rho_1, \rho_2, \cdots, \rho_m$ (assuming there are m counts of self-intersections) that represent null-homotopic loops. These words are called *relations* and we call the set $\mathbf{R} = \{\rho_1, \rho_2, \cdots, \rho_m\}$ the *relation set*. It can be easily noted that inverses and cyclic permutations of each ρ_i also corresponds to null-homotopic loops. We thus define the *symmetricized relation set*, $\overline{\mathbf{R}}$, as the set containing all the words in $\mathbf{R}$, all their inverses, and all cyclic permutation of each of those.

Let the normal subgroup of G generated by $\overline{\mathbf{R}}$ be $N = \{"\alpha_1 \rho_{i_1} \alpha_1^{-1} \ \alpha_2 \rho_{i_2} \alpha_2^{-1} \cdots$
$\alpha_\kappa^{-1} \alpha_\kappa \rho_{i_\kappa} \cdots" \mid \alpha_k \in G, \ \rho_{i_k} \in \overline{\mathbf{R}}\} = <\overline{\mathbf{R}}^G>$ (normal closure of $\overline{\mathbf{R}}$ in G). It is easy to observe that a closed loop, γ, in $X = \mathbb{R}^3 - \mathcal{O}$, has a word that is an element of N iff it is null-homotopic. Due to a more general version of the Van Kampen's theorem [Hatcher, 2001], the fundamental group of X is the quotient group, $\pi_1(X) = G/N = <\mathbf{U} \mid \mathbf{R}>$ — the group in which, under the quotient map, elements of N are mapped to the identity element. This is summarized and generalized in the following proposition.

Proposition 2 *Given a D-dimensional manifold (possibly with boundary), X, suppose $U_1, U_2, \cdots U_n$ are $(D-1)$-dimensional orientable sub-manifolds (not necessarily smooth and possibly with boundaries) such that $\partial U_i \subseteq \partial X \cup \bigcup_{j \neq i} U_j$ and*

1. a) The intersections of U_i and U_j, if non-empty, are transverse (hence $(D-2)$-dimensional),

b) $\bigcap_{i=1}^{n} U_i = \emptyset$,

2. $X - \bigcup_{i=1}^{n} U_i$ is path connected and simply-connected, and,

3. $\pi_1(X - \bigcup_{i=1, i \neq j}^{n} U_i) \simeq \mathbb{Z}, \ \forall j = 1, 2, \cdots, n$,

For a loop in X, as before, one can construct a word as described in Construction 1. Let the set of all the letters constituting such words be $\mathbf{U}$.

For every $(D-2)$-dimensional intersection of the $U_i \cap U_j$, $i \neq j$, one can construct a loop in the tubular neighborhood of $U_i \cap U_j$ in X that links with $U_i \cap U_j$. Let the set of words corresponding to such loops be $\mathbf{R}$.

Then the fundamental group of X is isomorphic to $<\mathbf{U} \mid \mathbf{R}>$.

3.2 The Word Problem and Dehn Algorithm

Due to the discussion above, two trajectories, γ_1, γ_2, connecting x_s and x_e in the knot/link complement, X, belong to the same homotopy class iff the word $h(\gamma_1 \cup -\gamma_2) = h(\gamma_1) \circ h(\gamma_2)^{-1}$ belongs to $N = <\overline{\mathbf{R}}^G>$. This problem in group theory is known as the *word problem* [Epstein, 1992], and there are various algorithms,

each suitable for specific types of groups, for solving the word problem. We, in particular, will focus on a very simple algorithm due to Max Dehn [Lyndon and Schupp, 2001, Greendlinger and Greendlinger, 1986], which is applicable to a wide class of groups and their presentations.

Dehn's metric algorithm: Given a presentation of a group, $\pi_1 = <\mathbf{U} \mid \mathbf{R}>$, we construct the *symmetricized* relation set $\overline{\mathbf{R}}$ as described earlier. Given a cyclically reduced word, w, made up of letters (and their inverses) from $\mathbf{U}$, one checks for every element $\rho \in \overline{\mathbf{R}}$ if w and ρ share a common sub-words that is of length greater than $|\rho|/2$ ($|\rho|$ being the length of ρ). If they do (say, $\rho = \alpha\beta\gamma$, with β being a sub-word appearing in w, and $|\beta| > |\rho|/2$), we replace the sub-word with the shorter equivalent that one obtains by setting ρ to the identity element (*i.e.*, replace β by $\alpha^{-1}\gamma^{-1}$ in w). This process is repeated, and the algorithm terminates when no more such sub-words are found. The final word at which the algorithm terminates indicates if the initial word, w, is in N (whether it maps to the identity element in π_1).

This algorithm can be used in conjunction with search in G_h as before for finding optimal trajectories in different homotopy classes, with two vertices $(x, w_1), (x, w_2) \in V_h$ being the same iff $h(w_1) \circ h(-w_2)$ reduces to the empty word upon applying the Dehn's metric algorithm.

3.3 Guarantees of Dehn Algorithm

It's well known [Greendlinger and Greendlinger, 1986] that if Dehn algorithm terminates at the empty word, then $w \in N$. However, the converse is not necessarily true. One can derive several sufficient (and often highly restrictive) conditions on the presentation $< \mathbf{U} \mid \mathbf{R} >$ under which the converse holds [Lyndon and Schupp, 2001]. If, for a given presentation of a group the converse holds, we say that Dehn algorithm is *complete* for that presentation (or that the presentation is complete with respect to the Dehn algorithm, or that the word problem is solvable using the specific presentation and Dehn algorithm).

Due to the result of [Weinbaum, 1971], the Dehn presentation of the fundamental groups of the complement of a tame, alternating, prime knot is complete with respect to the Dehn algorithm. It is also known [Epstein, 1992] that automatic groups (including hyperbolic groups) have presentations that are complete with respect to Dehn algorithm.

4 Cylindrically-deleted Configuration Spaces

The previous results are limited to 3-dimensional spaces: one suspects that higher dimensions are more difficult. However, there are some classes of spaces for which optimal path-planning with homotopy constraints is still computable via a Dehn algorithm, independent of dimension. The following class of examples is inspired by robot coordination problems, in which individual agents with predetermined motion paths have to coordinate their motions so as to avoid collision.

Consider a collection of n graphs $(\Gamma_i)_1^n$, each embedded in a common workspace (usually $\mathbb{R}^2$ or $\mathbb{R}^3$) with intersections permitted. In the simplest case, each Γ_i will be homeomorphic to a closed interval, but more general graphs are permitted, such as roadmap approximations to a configuration space. On each Γ_i, a robot R_i with some particular fixed size/shape is free to move. Such motion may be Euclidean (by translation/rotation); more general motions are possible, so long as the region occupied by the robot R_i in the workspace is purely a function of location on Γ_i. A point in the product space $\prod_i \Gamma_i$ determines the locations of the n robots in the common workspace. Certain configurations are illegal, due to collisions. For example, if the robots are point-like, and each $\Gamma_i = \Gamma$ is identical, then the configuration space of n points on Γ is the cross product $\prod_i \Gamma_i$ minus the pairwise diagonal Δ. If the robots are given finite extent, then this system has a configuration space obtained by the graph product $\prod_i \Gamma_i$ minus an ϵ-neighborhood of the pairwise diagonal. However, more general types of collisions can be defined, say, if the robots are irregularly shaped and the graphs Γ_i are all different. In this most general case, the natural analogue of a configuration space is the coordination spaces of [Ghrist and LaValle, 2006].

The *coordination space* of this system is defined to be the space of all configurations in $\prod_i \Gamma_i$ for which there are no collisions – the geometric robots R_i have no overlaps in the workspace. Under the assumption that collisions between robots are pairwise-defined, the coordination space is *cylindrically deleted* and of the form

$$X = \left(\prod_{i=1}^{n} \Gamma_i\right) - \mathcal{O} \quad \text{where} \quad \mathcal{O} = \bigcup_{i<j}\left\{(x_k)_1^N \in \prod_{k=1}^{N}\Gamma_k \;:\; (x_i, x_j) \in \Delta_{i,j}\right\},$$

for some (open, "collision") sets $\Delta_{i,j} \subset \Gamma_i \times \Gamma_j$ where $1 \leq i < j \leq N$. In what follows, we assume that the $\Delta_{i,j}$ are sufficiently tame (e.g., semialgebraic) so as to avoid issues of non-finitely-generated π_1. Given (internal, path-) metrics on each Γ_i, the coordination space X inherits a locally-Euclidean metric on products of edges in the graphs. Such X are complete path-spaces and thus the problem of geodesics is well-posed. Their fundamental groups can be (highly) nontrivial, depending on the obstacle set $\mathcal{O}$. However, finding optimal paths subject to homotopy classes is still computable. To that end one can construct the subspaces $U_i \subset X$ of co-dimension 1, and the relation set $\mathbf{R}$, and use them to design complete homotopy invariants as before. We do not discuss the explicit construction of the U_i's for cylindrically-deleted coordination spaces in this paper, but provide the following theorem on solvability of the word problem in such spaces.

Theorem 1 *Any compact cylindrically-deleted coordination space X admits a Dehn algorithm for π_1.*

> *Proof.* Any such X is realized as a Hausdorff limit of cubcial complexes which were shown in [Ghrist and LaValle, 2006, Thm 4.4] to be nonpositively-curved and to stabilize in π_1 by tameness. All nonpositively-curved piecewise-Euclidean cube complexes have fundamental groups which are, by a famous result of Niblo-Reeves [Niblo and Reeves, 1998], *biautomatic*. Biautomatic groups all admit a Dehn algorithm (specifically, there is a quadratic isoperimetric inequality) [Epstein, 1992].
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> $\square$

It is worth noting that ℓ_2-shortest paths are perhaps not the most natural optimization for coordination spaces. It would be interesting to consider other (ℓ_1, ℓ_∞) pointwise norms.

In the next section we consider point robots navigating on a plane. The configuration space of each robot is the Euclidean plane. For such a configuration space for the individual robots, the result of Theorem 1 may not hold, since the fundamental group of the coordination space may not be biautomatic. Nevertheless, we can construct a presentation of the fundamental group, which we do, and still apply Dehn metric algorithm to it, although the Dehn algorithm may not be complete for the proposed presentation.

4.1 Presentation of the Fundamental Group of a Cylindrically Deleted Coordination Space for Point Robots Navigating on a Plane

We consider point robots navigating on a plane. Thus, in this case Γ_i, the configuration space of the ith robot, is the Euclidean plane coordinatized by (x_i, y_i). A collision set $\Delta_{i,j} = \{(x_i, y_i, x_j, y_j) \mid x_i = x_j, y_i = y_j\} \subset \Gamma_i \times \Gamma_j \simeq \mathbb{R}^4$, $1 \leq i < j \leq N$, is a 2-dimensional hyperplane embedded in the joint configuration space of the robots i and j.

The joint configuration space of N robots (with the collision sets included) is $\overline{\Gamma} = \prod_{k=1}^{N}\Gamma_k \simeq \mathbb{R}^{2N}$. The "cylindrical" obstacles in this configuration space created due to $\Delta_{i,j}$ are thus

$$\mathcal{O}_{i,j} = \Delta_{i,j} \times \left(\prod_{k \neq i,j}\Gamma_k\right) = \{(x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid x_i = x_j, y_i = y_j\} \subset \overline{\Gamma},$$

which are co-dimension 2 subspaces (hyperplanes) embedded in the $2N$ dimensional joint configuration space. Thus the coordination space is $X = \overline{\Gamma} - \bigcup_{1 \leq i < j \leq N} \mathcal{O}_{i,j}$.

4.1.1 Design of co-dimension 1 manifolds, U_*:

As before, we are interested in constructing a set of $(2N - 1)$-dimensional sub-manifolds, $\{U_\alpha\}$, in X such that removing all but one of these sub-manifolds will give us a space with fundamental group isomorphic to $\mathbb{Z}$. This would let us apply the generalized Van Kampen theorem as before by allowing us to construct words

based on transverse intersection of paths with the surfaces, U_* (which would be of co-dimension 1 in X). We outline a general construction, and then show how that can be specialized for $N = 3$.

Consider a single $(2N - 2)$-dimensional obstacle $\mathcal{O}_{i,j}$, which is a hyperplane of co-dimension 2 in $\overline{\Gamma}$. Thus the homotopy group of $Y_{i,j} = \overline{\Gamma} - \mathcal{O}_{i,j}$ is isomorphic to $\mathbb{Z}$ and is generated by a 1-dimensional loop that *links* with $\mathcal{O}_{i,j}$ in this space. Our first construction corresponding to the obstacle $\mathcal{O}_{i,j}$ is thus the following half-space:

$$\mathcal{U}_{i,j} = \{(x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid x_i = x_j, y_i < y_j\}$$

This a $(2N - 1)$-dimensional (co-dimension 1 in $\overline{\Gamma}$) half-hyperplane with $\mathcal{O}_{i,j}$ at its boundary. However, for another pair of indices, $(i', j'), 1 \leq i' < j' \leq N$, the obstacle $\mathcal{O}_{i',j'}$ in general intersects $\mathcal{U}_{i,j}$ in a $(2N - 3)$-dimensional half-hyperplane, and the space $\mathcal{U}_{i,j}$ intersects $\mathcal{U}_{i,j}$ in a $(2N - 2)$-dimensional half-hyperplane. In particular,

$$\mathcal{U}_{i,j} \cap \mathcal{O}_{i',j'} = \{(x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid x_i = x_j, x_{i'} = x_{j'}, y_i < y_j, y_{i'} = y_{j'}\}$$

is of co-dimension 2 in $\mathcal{U}_{i,j}$, and,

$$\mathcal{U}_{i,j} \cap \mathcal{U}_{i',j'} = \{(x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid x_i = x_j, x_{i'} = x_{j'}, y_i < y_j, y_{i'} < y_{j'}\}$$

is of co-dimension 1 in $\mathcal{U}_{i,j}$ (a half-hyperplane) and is non-empty. This is schematically shown in Figure 7(a).

Lemma 1 $X - \bigcup_{1 \leq i < j \leq N} \mathcal{U}_{i,j} = \overline{\Gamma} - \bigcup_{1 \leq i < j \leq N} \mathcal{O}_{i,j} - \bigcup_{1 \leq i < j \leq N} \mathcal{U}_{i,j}$ *is path connected.*

> *Sketch of proof.* Suppose $\mathbf{p}^0 = (x_1^0, y_1^0, x_2^0, y_2^0, \cdots, x_N^0, y_N^0), \mathbf{p}^1 = (x_1^1, y_1^1, x_2^1, y_2^1, \cdots, x_N^1, y_N^1) \in X$. If $x_i^0 < x_j^0$, but $x_i^1 \geq x_j^1$, then whichever path, $t \mapsto \mathbf{p}^t, t \in [0, 1]$, is chosen to connect $\mathbf{p}^0$ and $\mathbf{p}^1$, there will be a τ for which $x_i^\tau = x_j^\tau$. However, if at this point, if $y_i^\tau \leq y_j^\tau$, then the path will be intersecting $\mathcal{O}_{i,j}$ or $\mathcal{U}_{i,j}$. Clearly, this can be prevented by simply altering the y_i^t and y_j^t in a small neighborhood of $t = \tau$, without altering any other coordinate and hence the alteration itself not leading to intersection with any other $\mathcal{U}_{i',j'}$. Thus an arbitrary pair of points in X an be connected using a path that does not intersect any of the $\mathcal{U}_{i,j}$'s.
>
> $\square$

Lemma 2 *Any loop in $\overline{\Gamma}$ linked only with $\mathcal{O}_{i,j}$ can be homotoped into any other loop linked only with $\mathcal{O}_{i,j}$, through a sequence of loops linked to $\mathcal{O}_{i,j}$,*

   i. *without intersecting $\mathcal{O}_{i',j'}$.*

   ii. *not without intersecting $\mathcal{U}_{i',j'}$ if either $i = i'$, $j' < j$ or $i < i'$, $j = j'$, otherwise without intersecting $\mathcal{U}_{i',j'}$.*

> *Sketch of proof.* The proof is based on being able to construct, or an obstruction to constructing, a homotopy satisfying certain properties between *small* loops (in a tubular neighborhood of $\mathcal{O}_{i,j}$) lying in a plane transverse to $\mathcal{O}_{i,j}$ and linking to it.
>
> *Case I: Distinct i, j, i' and j':*
>
> We first consider the case when i, j, i' and j' are all distinct. Consider a 2-dimensional affine plane transverse to $\mathcal{O}_{i,j}$ described by $\mathcal{P}_{i,j}(c_*, d_*) = \{(x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid x_i + x_j = c_{ij}, y_i + y_j = d_{ij}, x_k = c_k, y_k = d_k, \forall k \neq i, j\}$ (where c_* and d_* refers to the set of parameters describing the plane), and coordinatized by $X_{ij} = x_i - x_j$ and $Y_{ij} = y_i - y_j$ (so that $\mathcal{O}_{i,j}$ intersects the plane at its origin).
>
> The intersection of this plane with $\mathcal{O}_{i',j'}$ is, in general, empty except for carefully chosen values for the parameters (in particular, for parameters such that $c_{i'} = c_{j'}, d_{i'} = d_{j'}$), when they intersect over the entire $\mathcal{P}_{i,j}(c_*, d_*)$. Likewise, the intersection of this plane with $\mathcal{U}_{i',j'}$ is, in general, empty except when the parameters are such that $c_{i'} = c_{j'}, d_{i'} < d_{j'}$), when, again, they intersect at the entire $\mathcal{P}_{i,j}(c_*, d_*)$.
>
> Given two such affine planes, $\mathcal{P}_{i,j}(c_*^0, d_*^0)$ and $\mathcal{P}_{i,j}(c_*^1, d_*^1)$, for two different sets of parameters, one can easily choose a path, $t \mapsto (c_*^t, d_*^t), t \in [0, 1]$, through the parameter space avoiding $c_{i'}^t = $

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$c_{j'}^t, d_{i'}^t = d_{j'}^t$ simultaneously at any t. This gives a homotopy for a loop in $\mathcal{P}_{i,j}(c_*^0, d_*^0)$ around the origin (linking with $\mathcal{O}_{i,j}$) to a loop in $\mathcal{P}_{i,j}(c_*^1, d_*^1)$ around the origin, without intersecting $\mathcal{O}_{i',j'}$.

Similarly, it is possible to choose the path such that $c_{i'}^t = c_{j'}^t$ and $d_{i'}^t < d_{j'}^t$ does not happen simultaneously.

Case II: i, j, i' and j' not all distinct:

Next consider the case when i, j, i', j' are not distinct. Choose $1 \leq m < n < p \leq N$ to be the non-distinct indices such that either $i = i' = m < j' = n < j = p$, or $i = m < i' = n < j = j' = p$. Then the possible obstacles are $\mathcal{O}_{m,n}$, $\mathcal{O}_{n,p}$ and $\mathcal{O}_{m,p}$, with the indices of any pair of obstacle not all distinct.

Consider the plane $\mathcal{P}_{m,n}(c_*, d_*)$, on which $x_m + x_n = c_{mn}, y_m + y_n = d_{mn}$ and $x_p = c_p, y_p = d_p$. This intersects obstacle $\mathcal{O}_{n,p}$ at points where $x_n = x_p = c_p$ and $y_n = y_p = d_p$. This gives $x_m = c_{mn} - c_p, y_m = d_{mn} - d_p$. Thus, on the plane, the coordinates of the intersection point are $X_{mn} = c_{mn} - 2c_p, Y_{mn} = d_{mn} - 2d_p$. Once again, it is possible to choose the path (c_*^t, d_*^t) from (c_*^0, d_*^0) to (c_*^1, d_*^1) such that $c_{mn}^t - 2c_p^t$ and $d_{mn}^t - 2d_p^t$ are not simultaneously zero for any $t \in [0, 1]$. This argument also holds for the other two pairs of obstacles.

Again, using the chosen coordinates on $\mathcal{P}_{m,n}(c_*, d_*)$, $\mathcal{P}_{m,n}(c_*, d_*)$ intersects $\mathcal{U}_{n,p}$ at the ray $X_{mn} = c_{mn} - 2c_p, Y_{mn} > d_{mn} - 2d_p$, and intersects $\mathcal{U}_{m,p}$ at the ray $X_{mn} = c_{mn} - 2c_p, Y_{mn} < -(d_{mn} - 2d_p)$. Because the rays point in opposite directions along the X_{mn} axis, given the parameters (c_*^0, d_*^0) and (c_*^1, d_*^1), it is always possible to find the path (c_*^t, d_*^t) (in particular, d_p^t) such that the rays of intersection with $\mathcal{U}_{n,p}$ and $\mathcal{U}_{m,p}$ on $\mathcal{P}_{m,n}(c_*^t, d_*^t)$ does not pass through the origin (if $c_{mn}^0 - 2c_p^0$ and $c_{mn}^1 - 2c_p^1$ are of opposite signs, and if $c_{mn}^\tau - 2c_p^\tau = 0$ for some $\tau \in [0, 1]$, this can be achieved by choosing $d_{mn}^\tau - 2d_p^\tau > 0$). Similar argument holds for intersection of $\mathcal{P}_{n,p}(c_*, d_*)$ with $\mathcal{U}_{m,n}$ and $\mathcal{U}_{m,p}$.

However, $\mathcal{P}_{m,p}(c_*, d_*)$ intersects $\mathcal{U}_{m,n}$ at the ray $X_{mp} = 2c_n - c_{mp}, Y_{mp} < 2d_n - d_{mp}$, and $\mathcal{U}_{n,p}$ at the ray $X_{mp} = 2c_n - c_{mp}, Y_{mp} < -(2d_n - d_{mp})$. These rays point in the same direction. Clearly, given the parameters (c_*^0, d_*^0) and (c_*^1, d_*^1), if $2c_n^0 - c_{mp}^0$ and $2c_n^1 - c_{mp}^1$ are of opposite sign, it is not possible to find the path (c_*^t, d_*^t) such that neither of these rays of intersection do not pass through the origin in $\mathcal{P}_{m,p}(c_*^t, d_*^t)$.

$\square$

The consequence of the above Lemma is that the obstruction to fundamental group of $X_{m,p} = X - \cup_{(i',j') \neq (m,p), 1 \leq i' < j' \leq N} \mathcal{U}_{i',j'}$ being $\mathbb{Z}$ are the intersections of $\mathcal{U}_{m,p}$ with $\mathcal{U}_{m,n}$ and $\mathcal{U}_{n,p}$, for every n such that $m < n < p$. This leads us to split $\mathcal{U}_{m,p}$ by the the hyperplane at which it is intersected by $\mathcal{U}_{m,n}$ or $\mathcal{U}_{n,p}$, for every $m < n < p$. It can however be noted that for $1 \leq m < n < p \leq N$, $\mathcal{U}_{m,p} \cap \mathcal{U}_{m,n}$ and $\mathcal{U}_{m,p} \cap \mathcal{U}_{n,p}$ are subsets of the same $(2N - 2)$-dimensional hyperplane. Thus we have the following splitting for $\mathcal{U}_{m,p}$ only due to its intersection with $\mathcal{U}_{m,n}$ and $\mathcal{U}_{n,p}$:
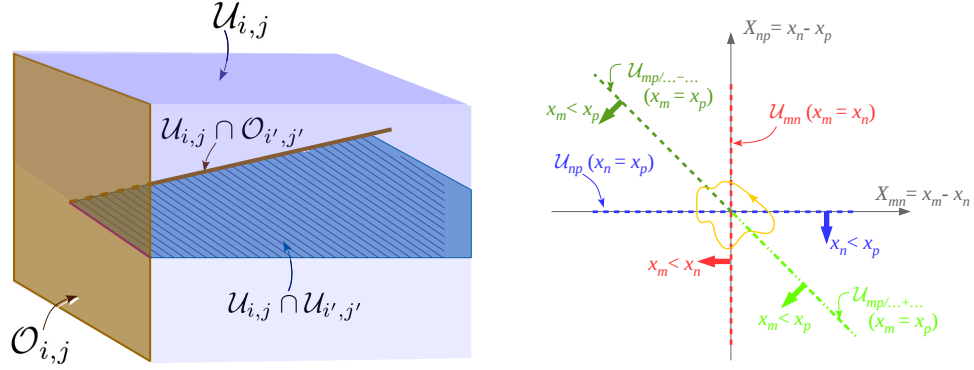
$$\mathcal{U}_{m,p/-} = \{(x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid x_m = x_p \leq x_n, \; y_m < y_p\}$$
$$\mathcal{U}_{m,p/+} = \{(x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid x_m = x_p \geq x_n, \; y_m < y_p\}$$

In general, we have partitions of the form

$$\mathcal{U}_{m,p/\sigma_{m+1}, \sigma_{m+2}, \cdots, \sigma_{p-1}} = \left\{ (x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid y_m < y_p, \left(\begin{array}{l} x_m = x_p \leq x_n \text{ if } \sigma_n = \text{`}-\text{'} \\ x_m = x_p \geq x_n \text{ if } \sigma_n = \text{`}+\text{'} \end{array}, \forall \, m < n < p \right) \right\} \tag{1}$$

Note that the σ's are indexed by integers from $m+1$ to $p-1$ just for convenience (instead of indexing using $1, 2 \cdots, p - m - 1$).

The following is a direct consequence

(a) Schematic illustration of $\mathcal{O}_{i,j}$, $\mathcal{U}_{i,j}$ and its intersection with $\mathcal{O}_{i',j'}, \mathcal{U}_{i',j'}$.

(b) Plane $\mathcal{P}_{m,n,p}(c_*, d_*)$ transverse to $\mathcal{U}_{m,p} \cap \mathcal{U}_{m,n} \cap \mathcal{U}_{n,p}$.

Figure 7: .

Corollary 1 *The fundamental group of*

$$X_{i,j/\sigma_{i+1},\sigma_{j+2},\cdots,\sigma_{j-1}} := X - \bigcup_{\substack{(i',j',\varsigma_{i'+1},\cdots,\varsigma_{j'-1}) \neq (i,j,\sigma_{i+1},\cdots,\sigma_{j-1}), \\ 1\leq i'<j'\leq N, \ \varsigma_k \in \{+,-\} \forall i'<k'<j'}} \mathcal{U}_{i',j'/\varsigma_{i'+1},\cdots,\varsigma_{j'-1}}$$

is isomorphic to $\mathbb{Z}$.

Thus, the $(2N-1)$ dimensional spaces, $\mathcal{U}_{m,p/\sigma_{m+1},\sigma_{m+2},\cdots,\sigma_{p-1}}$, where $1 \leq i < j \leq N$, and σ_k is either '+' or '−', satisfy the conditions of Proposition 2. The set **U** is freely generated by letters, $u_{m,p/\sigma_{m+1},\sigma_{m+2},\cdots,\sigma_{p-1}}$, corresponding to each of these spaces.

4.1.2   The Relation Set:

The relation set, **R**, contains the following types of words:

i. Due to the intersection of $\mathcal{U}_{m,p}$, $\mathcal{U}_{m,n}$ and $\mathcal{U}_{n,p}$ ($m < n < p$): $\mathcal{U}_{m,p}$, $\mathcal{U}_{m,n}$ and $\mathcal{U}_{n,p}$ intersect at a common $(2N-2)$-dimensional hyperplane on which $x_m = x_n = x_p$, and for $y_m < y_n < y_p$. Let $\mathcal{P}_{m,n,p}(c_*, d_*)$ be a plane transverse to this intersection, coordinatized by $X_{mn} = x_m - x_n$ and $X_{np} = x_n - x_p$, so that $\mathcal{U}_{m,p} \cap \mathcal{U}_{m,n} \cap \mathcal{U}_{n,p}$ intersects it at the origin on this plane. A loop around the origin in $\mathcal{P}_{m,n,p}(c_*, d_*)$ (Figure 7(b)) has the following word:

$$``u_{m,n/\alpha_{m+1},\cdots,\alpha_{n-1}} \cdot u_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(1)}=-,\cdots,\sigma_{p-1}} \cdot u_{n,p/\beta_{n+1},\cdots,\beta_{p-1}} \cdot$$
$$u_{m,n/\alpha_{m+1},\cdots,\alpha_{n-1}}^{-1} \cdot u_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(2)}=+,\cdots,\sigma_{p-1}}^{-1} \cdot u_{n,p/\beta_{n+1},\cdots,\beta_{p-1}}" \tag{2}$$

for any choice of signs $\alpha_a, \beta_b \in \{+,-\}$, $m < a < n, n < b < p$, and $\sigma_k = \begin{cases} \alpha_k, \text{ if } m < k < n \\ \beta_k, \text{ if } n < k < p \end{cases}$ (in order to ensure non-empty intersection). Note that the sign value of σ_n in the second letter of the word above is '−', while it is '+' in the fifth letter.

It can be noted that there exists non-empty intersection $\mathcal{U}_{m,p} \cap \mathcal{U}_{m,n}$ and $\mathcal{U}_{m,p} \cap \mathcal{U}_{n,p}$ that are outside the aforementioned common intersection of the three (the former when $y_m < y_n, y_m < y_p \leq y_n$ and the later when $y_n < y_p, y_n \leq y_m < y_p$). These intersections result in the following additional words in the relation set:

$$``u_{m,n/\alpha_{m+1},\cdots,\alpha_{n-1}} \cdot u_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(1)}=-,\cdots,\sigma_{p-1}} \cdot$$
$$u_{m,n/\alpha_{m+1},\cdots,\alpha_{n-1}}^{-1} \cdot u_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(2)}=+,\cdots,\sigma_{p-1}}^{-1} \cdot \tag{3}$$

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and

$$
\text{``} \begin{array}{ll} u_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(1)}=-,\cdots,\sigma_{p-1}} \cdot & u_{n,p/\beta_{n+1},\cdots,\beta_{p-1}} \cdot \\ u_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(2)}=+,\cdots,\sigma_{p-1}}^{-1} \cdot & u_{n,p/\beta_{n+1},\cdots,\beta_{p-1}} \end{array} \text{''} \tag{4}
$$

ii. Due to the intersection of $\mathcal{U}_{i,j}$ and $\mathcal{U}_{i',j'}$ (with $i \leq i'$), where i, j, i', j' are all distinct: Without loss of generality, assume either $i < i'$ or $i = i', j < j'$. $\mathcal{U}_{i,j}$ and $\mathcal{U}_{i',j'}$ intersect at a $(2N-2)$-dimensional hyperplane on which $x_i = x_j, x_{i'} = x_{j'}$. The letters corresponding to these spaces simply commute. Thus we have the words:

$$
\text{``} \begin{array}{ll} u_{i,j/\sigma_{i+1},\cdots,\sigma_{j-1}} \cdot & u_{i',j'/\gamma_{i'+1},\cdots,\gamma_{j'-1}} \cdot \\ u_{i,j/\sigma_{i+1},\cdots,\sigma_{j-1}}^{-1} \cdot & u_{i',j'/\gamma_{i'+1},\cdots,\gamma_{j'-1}}^{-1} \end{array} \text{''} \tag{5}
$$

for all $\sigma_k, \gamma_l \in \{+, -\}$

The relation set, $\mathbf{R}$, thus consists of all the words of the forms described in (2), (3), (4) and (5).

4.1.3 Explicit Example for $N = 3$

We consider the simple, yet non-rival case of $N = 3$ (coordination space of 3 robots navigating on a plane). The letters in $\mathbf{U}$ are $u_{1,2}$, $u_{2,3}$, $u_{1,3/+}$ and $u_{1,3/-}$ corresponding to respectively crossing of the manifolds $\mathcal{U}_{1,2} = \{\mathbf{p} \mid x_1 = x_2, y_1 < y_2\}$, $\mathcal{U}_{2,3} = \{\mathbf{p} \mid x_2 = x_3, y_2 < y_3\}$, $\mathcal{U}_{1,3/+} = \{\mathbf{p} \mid x_1 = x_3 > x_2, y_1 < y_3\}$ and $\mathcal{U}_{1,3/-} = \{\mathbf{p} \mid x_1 = x_3 < x_2, y_1 < y_3\}$. The relation set consists of the following words:

$$
\mathbf{R} = \{ \; u_{1,2} \; u_{1,3/-} \; u_{2,3} \; u_{1,2}^{-1} \; u_{1,3/+}^{-1} \; u_{2,3}^{-1},
$$
$$
u_{1,2} \; u_{1,3/-} \; u_{1,2}^{-1} \; u_{1,3/+}^{-1},
$$
$$
u_{1,3/-} \; u_{2,3} \; u_{1,3/+}^{-1} \; u_{2,3}^{-1} \; \}
$$

For simplicity, define $a = u_{1,2}$, $b = u_{1,3/+}$, $c = u_{1,3/-}$, $d = u_{2,3}$. Rewriting the relation set, $\mathbf{R} = \{acda^{-1}bd^{-1}, aca^{-1}b, cdbd^{-1}\}$. This can be easily shown (by isolating b by setting the last relation to identity, and substituting it in the other two relations) to be isomorphic to the group $< a, c, d \mid acd(dac)^{-1}, cda(dac)^{-1} >$. Which in turn (using substitution $p = cd, q = dac$) can be shown to be isomorphic to the group $< a, p, q \mid ap(pa)^{-1}, aq(qa)^{-1} >$ (a commutes with both p and q, but p and q themselves do not commute). This is clearly the group $\mathbb{Z} \times (\mathbb{Z} * \mathbb{Z})$. In fact, simply using geometric arguments, it is easy to verify that the homotopy type of the coordination space of 3 robots on a plane is indeed that of $\mathbb{S}^1 \times (\mathbb{S}^1 \vee \mathbb{S}^1)$ (see [Arslan et al., 2016] for example).

5 Simulation Results

5.1 Knot and Link Complements

Given obstacles $\mathcal{O} \subset \mathbb{R}^3$, and their "skeletons" (1-dimensional homotopy equivalents), $O \subseteq \mathcal{O}$ as polygons in $\mathbb{R}^3$, we first choose a projection map, $p : \mathbb{R}^3 \to \mathbb{R}^2$, for the knot/link diagram. With this information, we implemented the automated construction of the surfaces, U_i, for the Dehn presentation of the knot/link complement, and the symmetricized relation set, $\overline{\mathbf{R}}$, by computing the self-intersections in $p(O)$. We then used a uniform cubical discretization of $\mathbb{R}^3 - \mathcal{O}$ to construct the graph G as a discrete representation of the free space, and in the h-augmented graph, G_h, we find trajectories from $(x_s, \text{``''})$ to $(x_g, *)$. We then employ a curve shortening algorithm to shorten the obtained trajectories. All our implementations were done in C++ programming language and visualization were done using OpenGL. The program ran on a laptop running on a Intel i7-4500U processor @ 1.80GHz with 8 GB memory.

Figure 8(a) shows results in the complement of a trefoil knot. The inset figure shows the surfaces, U_i, used for Dehn presentation. The graph G was constructed out of uniform $100 \times 100 \times 100$ cubical discretization of the environment. The entire computation (computation of the surfaces, the symmetricized relation set $\mathbf{R}$, and computation of the 5 shortest trajectories) required about 8.1 s. Likewise, Figure 8(b) shows results in the complement of a simple Hopf link, with the same discretization of the environment, and total computation time of about 8.2 s. Figure 8(c) shows a much more complex obstacle involving a torus knot linked to a genus-2 obstacle, and the entire computation of 20 trajectories took about 2.6 s.

(a) 5 trajectories in a trefoil knot complement.

(b) 5 trajectories in a Hopf link complement.

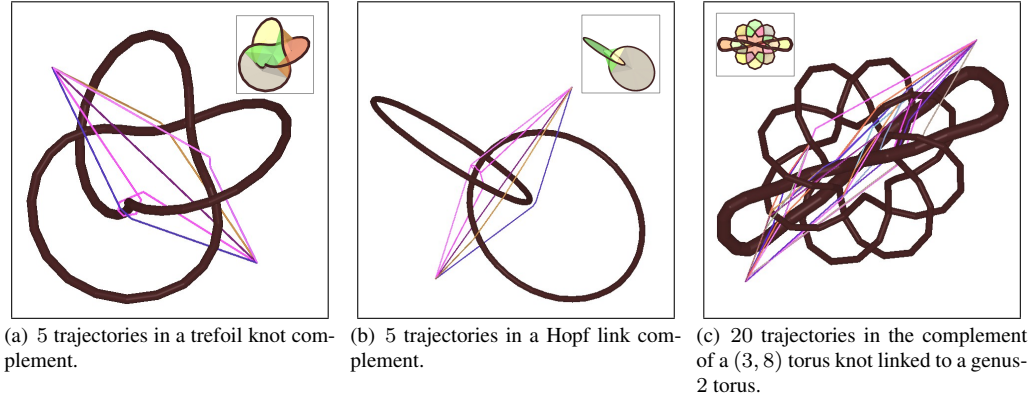(c) 20 trajectories in the complement of a $(3, 8)$ torus knot linked to a genus-2 torus.

Figure 8: Optimal trajectories (in discrete graph representation, followed by curve shortening) in different homotopy classes in complements of knots and links. Insets show the surfaces, U_i.

5.2   Coordination Space of 3 Robots Navigating on a Plane

Similar to the method described in Section 2.4, we implemented the graph search based approach for finding optimal paths in the coordination space of 3 robots navigating on a planar region. We constructed a graph using a uniform hyper-cubical discretization of the configuration space, $\overline{\Gamma} = \mathbb{R}^6$, placing a vertex in each cell, and establishing edges between neighboring cells that correspond to robot motions parallel to the coordinate axes. The optimality criteria in the A* search was chosen to be the sum of the total lengths of the paths (which, because of the chosen discretization, is the length due to the Manhattan metric on the plane) traversed by the robots. Figure 9 shows paths found in 5 different homotopy classes for the coordination space of 3 robots navigating on a plane. With the plane discretized into 7×7 grid, and the degree of each vertex in the corresponding joint configuration space being 124 (allowing each robot to move north, south, east or west, or stay in place), the computation of these paths 5 different homotopy classes took about $31.7\,\text{s}$.

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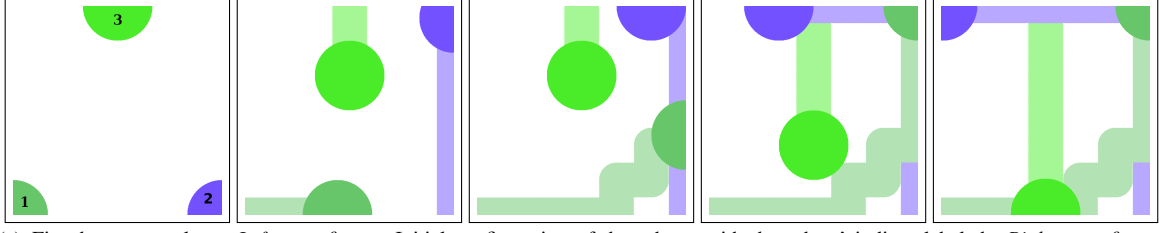
(a) First homotopy class. *Left-most figure:* Initial configuration of the robots, with the robots' indices labeled. *Right-most figure:* Final configuration of the robots. Intermediate figures illustrate the paths taken by the robots. Word corresponding to this class is "$u_{1,3/-}^{-1} \cdot u_{1,2}^{-1}$"
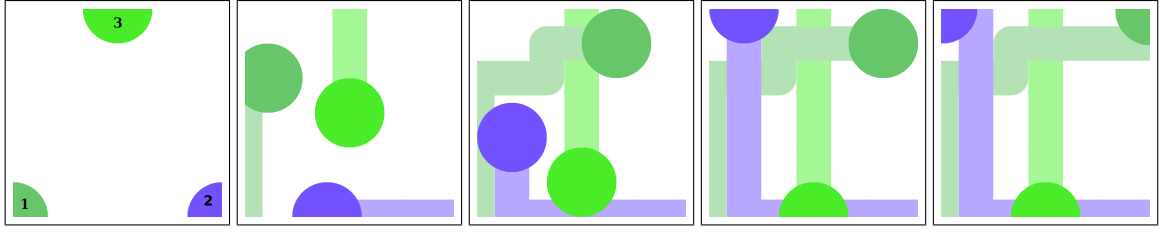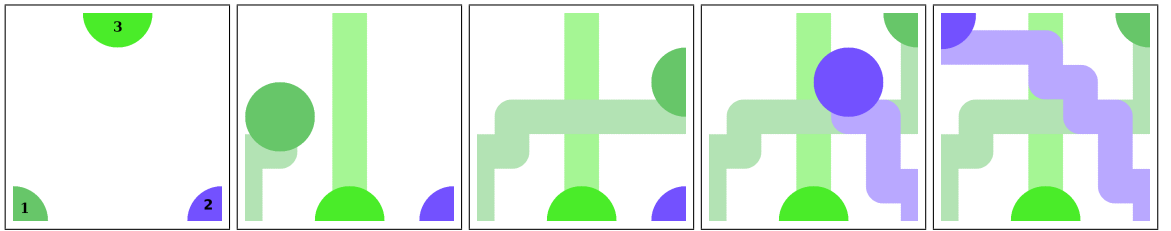
(b) Second homotopy class corresponding to word "$u_{2,3}$".
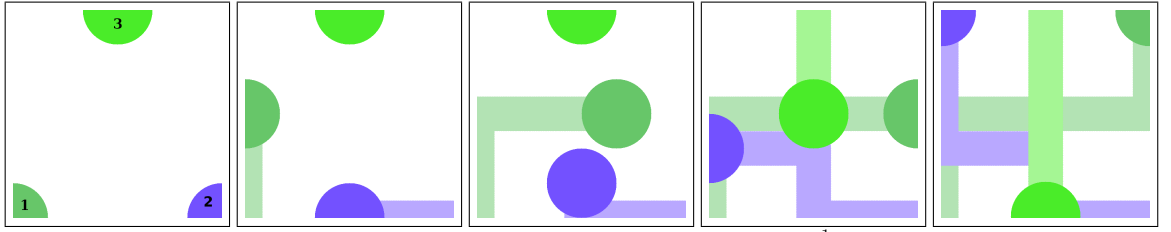
(c) Third homotopy class corresponding to word "$u_{2,3} \cdot u_{1,3/+}^{-1}$"
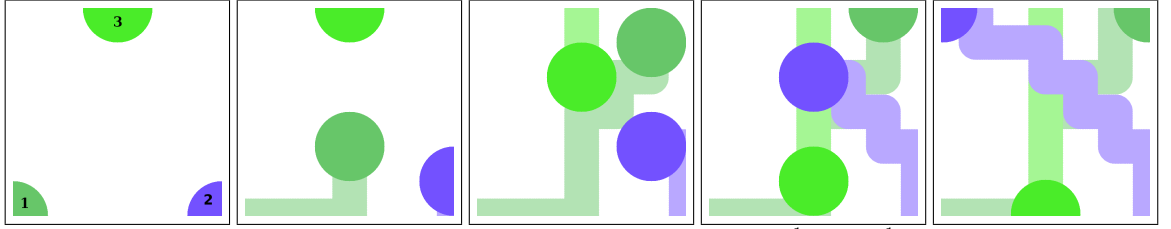
(d) Fourth homotopy class corresponding to word "$u_{2,3} \cdot u_{1,3/+}^{-1} \cdot u_{2,3}^{-1}$"

(e) Fifth homotopy class corresponding to word " "

Figure 9: Paths in 5 different homotopy classes in the coordination space of 3 robots navigating on a plane for an initial configuration to a final goal configuration. The classes found are in ascending order of the sum of the lengths of the paths of the three robots (where length is induced by the Manhattan metric on the plane).

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