H-signature for General Euclidean Spaces Punctured by Singularity Manifolds

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I. H-signature for 2 and 3 dimensions

A. 2-dimensional case

Recall we defined the function $\mathcal{H}_2 : C_1(\mathbb{C}) \rightarrow \mathbb{C}^N$ such that

$$\mathcal{H}_2(\tau) = \int_\tau \mathcal{F}(z)\mathrm{d}z$$

where,

$$\mathcal{F}(z) = \begin{bmatrix} \frac{f_1(z)}{z-\zeta_1} \\ \frac{f_2(z)}{z-\zeta_2} \\ \vdots \\ \frac{f_N(z)}{z-\zeta_N} \end{bmatrix}$$

with $f_l,\ l = 1, 2, \cdots, N$ being analytic functions over entire $\mathbb{C}$ such that $f_l(\zeta_l) \neq 0,\ \forall l$

B. 3-dimensional case

For 3 dimensions, $\mathcal{H}_3 : C_1(\mathbb{R}^3) \rightarrow \mathbb{R}^M$ is such that

$$\mathcal{H}_3(\tau) = [h_1(\tau),\ h_2(\tau),\ \ldots,\ h_M(\tau)]^T$$

where,

$$h_i(\tau) = \int_\tau \mathbf{B}_i(\mathbf{l}) \cdot \mathrm{d}\mathbf{l}$$

with

$$\mathbf{B}_i(\mathbf{r}) = \frac{1}{4\pi} \int_{S_i} \frac{(\mathbf{x} - \mathbf{r}) \times \mathrm{d}\mathbf{x}}{\|\mathbf{x} - \mathbf{r}\|^3}$$

II. Extension to Higher Dimensions

Now we attempt to generalize the formulae for H-signature to arbitrary dimension D. We provide a simplified derivation in terms of exterior calculus and the Stokes theorem [6, 4, 5]. This analysis is reminiscent of a more general treatment that we are currently investigating [1].

We consider the D-dimensional Euclidean space $\mathbb{R}^D$ with $(D-2)$-dimensional compact, closed (boundaryless), locally contractible and orientable sub-manifolds (which we call *singularity manifolds*), $S_1, S_2, \cdots, S_m$. The singularity manifolds are the analogs of the "representative points" in the 2-dimensional case or the "skeletons" in the 3-dimensional case. In the discussions that follows, we will use subscripts $1, 2, \cdots, D$, to denote the different components of a vector quantity. For example, $\mathbf{y} = [y_1, y_2, \cdots, y_D]^T$.

Let us define the vector function,

$$\mathcal{G}(\mathbf{y}) = \frac{1}{4\pi} \frac{\mathbf{y}}{(y_1^2 + y_2^2 + \cdots + y_D^2)^{3/2}} \tag{1}$$

This is the *Green's Function* in D dimensions for the Gradient operator [2, 3], and it is a well-known result that

$$\sum_{k=1}^{D} \frac{\partial \mathcal{G}_k}{\partial y_k}\Big|_{\mathbf{y}} = \delta(\mathbf{y}) \tag{2}$$

where $\delta(\cdot)$ is the *Dirac Delta Distribution* in $\mathbb{R}^D$ [2], and $\mathcal{G}_k$ represent the k^{th} component of $\mathcal{G}$.

We now recall the definition of *exterior derivative* [6, 4] of a differential form ϕ [6, 4]: $\mathrm{d}(\phi) = \sum_p \frac{\partial \phi}{\partial y_p} \wedge \mathrm{d}y_p$. By multiplying both sides of equation (2) with the differential D-form, $\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \cdots \wedge \mathrm{d}y_D$, and noting that $\mathrm{d}y_i \wedge \mathrm{d}y_i = 0$, we can rewrite the equation as,

$$\begin{aligned} &\left(\frac{\partial \mathcal{G}_1}{\partial y_1} + \frac{\partial \mathcal{G}_2}{\partial y_2} + \cdots + \frac{\partial \mathcal{G}_D}{\partial y_3} \right) \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \cdots \wedge \mathrm{d}y_D \\ &\qquad = \delta(\mathbf{y})\ \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \cdots \wedge \mathrm{d}y_D \\ &\Rightarrow\ \mathrm{d}\left(\textstyle\sum_{k=1}^{D} \left((-1)^{k+1} \mathcal{G}_k\ \bigwedge_{l \neq k} \mathrm{d}y_l \right) \right) \Big|_{\mathbf{y}} \\ &\qquad = \delta(\mathbf{y})\ \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \cdots \wedge \mathrm{d}y_D \end{aligned} \tag{3}$$

where the operator "d" at the beginning of the left land side of the equation is the *exterior derivative*.

Now consider the single connected component of the singularity manifold, $\mathcal{S}$, as a $(D-2)$-dimensional manifold embedded in a D-dimensional Euclidean space, $\mathbb{R}^D_{\mathbf{x}}$ (the subscript is used to denote that this is the space where the singularity manifold is embedded). Now consider an arbitrary 2-dimensional surface, Ω, whose boundary is γ (the closed curve along which we perform the integration: the closed loop formed by $\tau_1 \sqcup -\tau_2$), embedded in a different D-dimensional Euclidean space, $\mathbb{R}^D_{\mathbf{r}}$. A point in $\mathbb{R}^D_{\mathbf{x}}$ is represented by $\mathbf{x}$, while that in $\mathbb{R}^D_{\mathbf{r}}$ is represented by $\mathbf{r}$ (Figure 1 illustrates this for 3 dimensions).

Now consider the D-dimensional manifold created by the Cartesian product $\Omega \times \mathcal{S}$ embedded in the $2D$-dimensional product space $\mathbb{R}^D_{\mathbf{r}} \times \mathbb{R}^D_{\mathbf{x}}$. Let us consider the tangent space

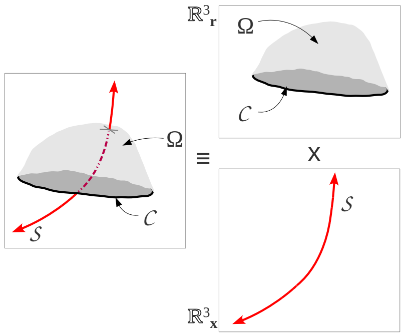

Fig. 1. We imagine the current carrying conductor, $\mathcal{S}$, and the integration curve, C, to be residing in two separte $\mathbb{R}^3$.

at any point on this manifold. Since it is a D dimensional vector space, out of the $2D$ differentials, $\{\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{x}\} \equiv \{\mathrm{d}r_1, \mathrm{d}r_2, \cdots, \mathrm{d}r_D, \mathrm{d}x_1, \mathrm{d}x_2, \cdots, \mathrm{d}x_D\}$, only D can be linearly independent when we stay constrained in the manifold $\Omega \times \mathcal{S}$. Moreover, since $\mathcal{S}$ is itself a $(D-2)$ dimensional manifold described by the coordinates $\mathbf{x}$, on $\Omega \times \mathcal{S}$ out of $\{\mathrm{d}x_1, \mathrm{d}x_2, \cdots, \mathrm{d}x_D\}$, only $(D-2)$ can be independent. Similarly, since Ω is 2-dimensional, out of $\{\mathrm{d}r_1, \mathrm{d}r_2, \cdots, \mathrm{d}r_D\}$, only 2 can be linearly independent on $\Omega \times \mathcal{S}$. As a consequence of these, the following identities hold on $\Omega \times \mathcal{S}$,

$$
\begin{aligned}
& \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_p} \wedge \mathrm{d}r_{i_{p+1}} \wedge \cdots \wedge \mathrm{d}r_{i_{p+q}} = 0, \\
& \qquad\qquad\qquad\qquad \text{whenever } p+q > D, \quad \text{and} \\
& \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_p} = 0, \quad \text{whenever } p > D-2, \quad \text{and} \\
& \mathrm{d}r_{i_1} \wedge \mathrm{d}r_{i_2} \wedge \mathrm{d}r_{i_3} = 0, \\
& \mathrm{d}r_{i_1} \wedge \mathrm{d}r_{i_2} \wedge \mathrm{d}r_{i_3} \wedge \mathrm{d}r_{i_4} = 0, \\
& \cdots \\
& \mathrm{d}r_{i_1} \wedge \mathrm{d}r_{i_2} \wedge \cdots \wedge \mathrm{d}r_{i_D} = 0
\end{aligned}
\tag{4}
$$

Setting $\mathbf{y} = \mathbf{r} - \mathbf{x}$, we will now integrate both sides of Equation (3) (which are differential D-forms) on the D-dimensional manifold $\Omega \times \mathcal{S}$. With the substitution $\mathbf{y} = \mathbf{r} - \mathbf{x}$, Equation (3) becomes,

$$
\begin{aligned}
& \mathrm{d}\left(\sum_{k=1}^{D} \left((-1)^{k+1} \mathcal{G}_k \bigwedge_{l \neq k} (\mathrm{d}r_l - \mathrm{d}x_l) \right) \right)\Bigg|_{\mathbf{r}-\mathbf{x}} \\
& \quad = \delta(\mathbf{r}-\mathbf{x})\ (\mathrm{d}r_1 - \mathrm{d}x_1) \wedge (\mathrm{d}r_2 - \mathrm{d}x_2) \wedge \cdots \wedge (\mathrm{d}r_D - \mathrm{d}x_D)
\end{aligned}
\tag{5}
$$

On $\Omega \times \mathcal{S}$ we can choose any one of the coordinate systems among $\mathbf{y}, \mathbf{x}$ or $\mathbf{p}$. Now we use the property of coordinate invariance of exterior derivative operator. That is, $\mathrm{d}(\phi) = \sum_p \frac{\partial \phi}{\partial y_p} \wedge \mathrm{d}y_p = \sum_p \frac{\partial \phi}{\partial x_p} \wedge \mathrm{d}x_p = \sum_p \frac{\partial \phi}{\partial r_p} \wedge \mathrm{d}r_p$. By choosing $\mathbf{r}$ as our preferred coordinate, we note that most of the terms in the left hand side of Equation (5) that contain differentials of the form as described in Equation (4) vanish identically on $\Omega \times \mathcal{S}$. Using this observation, and upon some simplification, the integration of (5) on $\Omega \times \mathcal{S}$ becomes,

$$
\begin{aligned}
& \int_{\Omega \times \mathcal{S}} \mathrm{d}\Big(\sum_{k=1}^{D} \sum_{\substack{j=1 \\ j \neq k}}^{D} (-1)^{k-j-1-\mathbf{is}(j<k)} \mathcal{G}_k(\mathbf{r}-\mathbf{x}) \\
& \qquad\qquad (\bigwedge_{l \neq j,k} \mathrm{d}x_l) \wedge \mathrm{d}r_j \Big) \quad = \\
& \qquad\qquad\qquad \int_{\Omega \times \mathcal{S}} \delta(\mathbf{y})\ \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \cdots \wedge \mathrm{d}y_D
\end{aligned}
\tag{6}
$$

where, for the right-hand-side of the equation we once again use the coordinates $\mathbf{y} = \mathbf{r} - \mathbf{x}$, and $\mathbf{is}(\mathit{cond.})$ is 1 if $(\mathit{cond.})$ is true, 0 otherwise.

Now, by using the Stoke's Theorem [6, 4], $\int_\Sigma \mathrm{d}\phi = \int_{\partial\Sigma} \phi$, where $\partial\Sigma$ is the boundary of the manifold Σ. However, since the singularity manifold, $\mathcal{S}$, does not have a boundary, $\partial(\Omega \times \mathcal{S}) = \partial(\Omega) \times \mathcal{S} = \gamma \times \mathcal{S}$. Thus, equation (6) becomes,

$$
\begin{aligned}
& \int_\gamma \int_{\mathcal{S}} \Big(\sum_{k=1}^{D} \sum_{\substack{j=1 \\ j \neq k}}^{D} (-1)^{k-j-1-\mathbf{is}(j<k)} \mathcal{G}_k(\mathbf{r}-\mathbf{x}) \\
& \qquad\qquad\qquad (\bigwedge_{l \neq j,k} \mathrm{d}x_l) \wedge \mathrm{d}r_j \Big) \\
& \qquad = \int_{\Omega \times \mathcal{S}} \delta(\mathbf{y})\ \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \cdots \wedge \mathrm{d}y_D \\
& \qquad = \begin{cases} \pm 1, & \text{if } \Omega \times \mathcal{S} \text{ contains the point } \mathbf{y} = 0 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{7}
$$

where the content of the right-hand-side follows directly from the property of *Dirac Delta Distribution*, and the sign depends on the orientation of $\Omega \times \mathcal{S}$. We note that the statement "$\Omega \times \mathcal{S}$ contains the point $\mathbf{y} = 0$" is equivalent to saying that there is no intersection between Ω and $\mathcal{S}$. That is, $\gamma = \partial\Omega$ is *null-homologous* in $(\mathbb{R} - \mathcal{S})$.

Thus, for general D dimensional Euclidean configuration space with $S_1, S_2, \cdots, S_M$ as $(D-2)$-dimensional homotopy equivalents of the obstacles such that each S_i is connected and closed (boundaryless), we define the H-signature of a trajectory τ as,

$$
\mathcal{H}_D(\tau) = \int_\tau \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_M \end{bmatrix}
\tag{8}
$$

where, ω_i are differential 1-forms defined as,

$$
\omega_i(\mathbf{r}) = \sum_{k=1}^{D} \sum_{\substack{j=1 \\ j \neq k}}^{D} U_j^k(\mathbf{r}; S_i)\ \mathrm{d}r_j
$$

with,

$$
\begin{aligned}
U_j^k(\mathbf{r}; S) \ &= \ (-1)^{k-j-1-\mathbf{is}(j<k)} \\
& \qquad \int_S \mathcal{G}_k(\mathbf{r}-\mathbf{x})\ \mathrm{d}x_1 \wedge \mathrm{d}x_2 \cdots \wedge \widehat{x_j, x_k} \wedge \cdots \wedge \mathrm{d}x_D
\end{aligned}
$$

where, the *hat* over x_j and x_k imply that those therms are not present in the differential.

It can be shown that this formula reduces to the formula for $\mathcal{H}_2$ when we set $D = 2$ and reduces to formula for $\mathcal{H}_3$ when we set $D = 3$. For a more in-depth analysis see [1].

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