Supplementary material: *h*-signature of a Non-looping Trajectory with Respect to an Infinite Straight Line Skeleton

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A. *Recall the definition of H-signature*

Definition 1 (*h*-Signature): Given an arbitrary trajectory, τ, in the 3-dimensional environment with M skeletons, we define the *H-signature* of τ to be the following M-vector,

$$\mathcal{H}(\tau) = [h_1(\tau),\ h_2(\tau),\ \ldots,\ h_M(\tau)]^T \tag{1}$$

where,

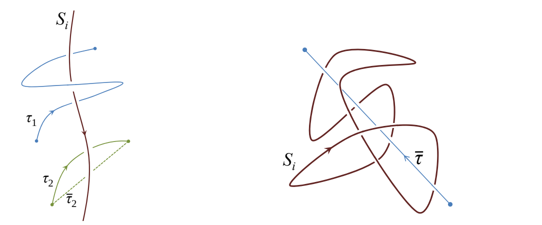
$$h_i(\tau) = \int_\tau \mathbf{B}_i(\mathbf{l}) \cdot \mathrm{d}\mathbf{l} \tag{2}$$

is defined in an analogous manner as the integral in Ampere's Law. In defining h_i, $\mathbf{B}_i$ is the *Virtual Magnetic Field* vector due to the unit current through skeleton S_i, $\mathbf{l}$ is the integration variable that represents the coordinate of a point on τ, and $d\mathbf{l}$ is an infinitesimal element on τ.

B. *"Looping" and "Non-loooping" Trajectories*

"Looping" of a trajectory around an obstacle (Figure 1(a)) is similar in essence to *non-Jordan* curves on two-dimensional planes. However in three dimensions their precise and universal definition is more difficult. One way of identifying one of the homotopy classes of trajectories (joining a given start and an end coordinate) that do not loop around a skeleton S_i is by joining the start and the end coordinates using a straight line segment (call it $\overline{\tau}$). Then the trajectories that are homotopic to $\overline{\tau}$ form a particular homotopy class of *non-looping trajectories* w.r.t. S_i (for example, in Figure 1(a), the homotopy class to which $\overline{\tau}_2$, and hence τ_2, belong are non-looping). However, for more complex obstacles (like knots), the notion of a *non-looping* trajectory being a straight line segment breaks down (See Figure 1(b)). In fact the notions of *looping* and *non-looping* is imprecise in such cases. In Section-C we show that for the special simple case when S_i is an infinitely long line, the component of the *h*-signature $h_i(\overline{\tau})$ for a line segment $\overline{\tau}$ lies between -1 and 1. We hence propose the following mathematical definition of a *non-looping* trajectory,

Definition 2 (Non-looping trajectory w.r.t. S_i): A trajectory τ is said to be *non-looping* w.r.t. S_i if $h_i(\tau) \in (-1, 1)$.



(a) Trajectory that loops around a skeleton & one that doesn't. In this figure $h_i(\tau_1) > 1$ and $0 < h_i(\tau_2) = h_i(\overline{\tau}_2) < 1$.

(b) In the most general case, it is difficult to precisely identify a *non-looping* homotopy class.

Fig. 1.

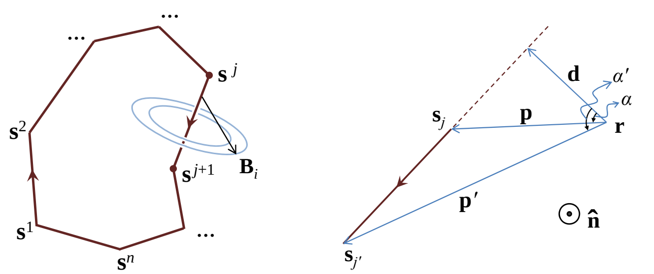
The value is in $[0, 1)$ if the trajectory goes around S_i in accordance with the *right-hand rule* with thumb pointing along the direction of the current in S_i. If the direction is opposite, the value lies in $(-1, 0]$.

This definition, in many cases, conform to our general intuition of *non-looping* trajectories. If another trajectory, τ', connecting the same start and end points as a non-looping trajectory τ, goes on the "other side of the obstacle" without looping around it, then $\tau \cup -\tau'$ forms a closed loop that encloses S_i. Then, $h_i(\tau \cup -\tau') = \pm 1 = \mathrm{sign}(h_i(\tau \cup -\tau'))$. But since, τ and $\tau \cup -\tau'$ goes around S_i in the same orientation, we have $\mathrm{sign}(h_i(\tau \cup -\tau')) = \mathrm{sign}(h_i(\tau))$. Again by property of line integration, $h_i(\tau \cup -\tau') = h_i(\tau) - h_i(\tau')$. Thus, $h_i(\tau') = h_i(\tau) - \mathrm{sign}(h_i(\tau))$. Thus we have the following definition.

Definition 3 (Complementary Homotopy Class): Given a trajectory τ that is *non-looping* w.r.t. all the skeletons in the environment (*i.e.* $h_i(\tau) \in (-1, 1)\ \forall\ i = 1, 2, \ldots, M$), we define the *Complementary Homotopy Class* of the homotopy class of τ to be the one for which the *h*-signature is $\mathcal{H}(\tau) - \mathrm{sign}(\mathcal{H}(\tau))$, where $\mathrm{sign}(\cdot)$ gives the vector of signs of the elements of its input vector.

C. *Computation of H-Signature for a line segment (*e.g* an Edge of $\mathcal{G}$)*

For all practical applications we assume that a skeleton of an obstacle, S_i, is made up of finite number (n_i) of line segments:



(a) A skeleton of an obstacle can be constructed or approximated such that it is made up of n line segments.

(b) Magnetic field at $\mathbf{r}$ due to the current in a line segment $\overrightarrow{\mathbf{s}_i^j\mathbf{s}_i^{j'}}$ can be computed analytically.

Fig. 2.

$S_i = \{\overrightarrow{\mathbf{s}_i^1\mathbf{s}_i^2},\ \overrightarrow{\mathbf{s}_i^2\mathbf{s}_i^3},\ \ldots,\ \overrightarrow{\mathbf{s}_i^{n_i-1}\mathbf{s}_i^{n_i}},\ \overrightarrow{\mathbf{s}_i^{n_i}\mathbf{s}_i^1}\}$ (Figure 2(a)). Thus, the integration in the Biot-Savert law can be split into summation of n_i integrations,

$$\mathbf{B}_i(\mathbf{r}) = \frac{1}{4\pi}\sum_{j=1}^{n_i}\int_{\overrightarrow{\mathbf{s}_i^j\mathbf{s}_i^{j'}}} \frac{(\mathbf{x}-\mathbf{r})\times \mathrm{d}\mathbf{x}}{\|\mathbf{x}-\mathbf{r}\|^3} \tag{3}$$

where $j' \equiv 1 + (j \mod n_i)$.

One advantage of this representation of skeletons is that for the straight line segments, $\overrightarrow{\mathbf{s}_i^j\mathbf{s}_i^{j'}}$, the integration can be computed analytically. Specifically, using a result from [1] (also, see Figure 2(b)),

$$\begin{aligned}\int_{\overrightarrow{\mathbf{s}_i^j\mathbf{s}_i^{j'}}} \frac{(\mathbf{x}-\mathbf{r})\times \mathrm{d}\mathbf{x}}{\|\mathbf{x}-\mathbf{r}\|^3} &= \frac{1}{\|\mathbf{d}\|}\left(\sin(\alpha')-\sin(\alpha)\right)\hat{\mathbf{n}} \\ &= \frac{1}{\|\mathbf{d}\|^2}\left(\frac{\mathbf{d}\times\mathbf{p}'}{\|\mathbf{p}'\|} - \frac{\mathbf{d}\times\mathbf{p}}{\|\mathbf{p}\|}\right)\end{aligned} \tag{4}$$

where, $\mathbf{d}, \mathbf{p}$ and $\mathbf{p}'$ are functions of $\mathbf{s}_i^j, \mathbf{s}_i^{j'}$ and $\mathbf{r}$ (Figure 2(b)), and can be expressed as,

$$\begin{gathered}\mathbf{p} = \mathbf{s}_i^j - \mathbf{r}\ , \quad \mathbf{p}' = \mathbf{s}_i^{j'} - \mathbf{r}\ , \\ \mathbf{d} = \frac{(\mathbf{s}_i^{j'} - \mathbf{s}_i^j)\times(\mathbf{p}\times\mathbf{p}')}{\|\mathbf{s}_i^{j'} - \mathbf{s}_i^j\|^2}\end{gathered} \tag{5}$$

We define and write $\mathbf{\Phi}(\mathbf{s}_i^j, \mathbf{s}_i^{j'}, \mathbf{r})$ for the RHS of Equation (4) for notational convenience. Thus we have,

$$\mathbf{B}_i(\mathbf{r}) = \frac{1}{4\pi}\sum_{j=1}^{n_i}\mathbf{\Phi}(\mathbf{s}_i^j, \mathbf{s}_i^{j'}, \mathbf{r}) \tag{6}$$

where, $j' \equiv 1 + (j \mod n_i)$.

Given an edge $e \in \mathcal{E}$, we can now compute the *h-signature*, $\mathcal{H}(e) = [h_1(e),\ h_2(e),\ \ldots,\ h_M(e)]^T$, where,

$$h_i(e) = \frac{1}{4\pi}\int_e \sum_{j=1}^{n_i}\mathbf{\Phi}(\mathbf{s}_i^j, \mathbf{s}_i^{j'}, \mathbf{l})\cdot \mathrm{d}\mathbf{l} \tag{7}$$

can be computed numerically.

Making use of the result from Equation (4), if the current carrying line segment stretches to infinity in both direction (*i.e.* it becomes a line), we have $\alpha' = \frac{\pi}{2}$ and $\alpha = -\frac{\pi}{2}$. The virtual magnetic field due to S_i at a point $\mathbf{r}$ becomes

$$\mathbf{B}_i = \frac{1}{4\pi}\frac{2\,\hat{\mathbf{n}}}{\|\mathbf{d}\|} = \frac{1}{2\pi}\frac{\hat{\mathbf{n}}}{\|\mathbf{d}\|} \tag{8}$$

Note that the contribution of the closing curve at infinity becomes zero in the above quantity.

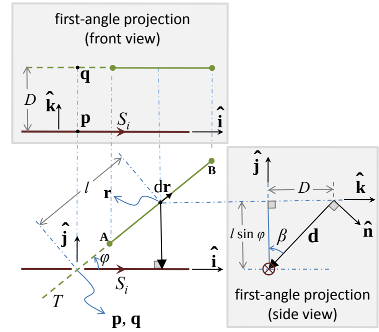


Fig. 3. An infinitely long skeleton and h-signature of a straight line segment.

Now consider the straight line segment trajectory $\overline{\tau} = \overline{\mathbf{r}_A\mathbf{r}_B}$. Let the line containing $\overline{\tau}$ (*i.e.* formed by extending $\overline{\tau}$ to infinity in both directions) be T (Figure 3). Consider the shortest distance between S_i and T and let it be D. Assuming S_i and T are not parallel, there is a unique point on each of these line ($\mathbf{p}$ and $\mathbf{q}$ respectively) that are closest and are separated by the distance D. The line segment joining the closest points, $\overline{\mathbf{pq}}$, is perpendicular to both S_i and T. The main diagram of Figure 3 shows the projection of S_i and T on a plane perpendicular to $\overline{\mathbf{pq}}$. Note that this plane (the plane of the paper) is parallel to both S_i and T, since it is perpendicular to $\overline{\mathbf{pq}}$.

We define an orthonormal coordinate system with unit vectors $\hat{\mathbf{i}}$ pointing along S_i in the direction of the current, and unit vector $\hat{\mathbf{k}}$ pointing along $\overline{\mathbf{pq}}$. Using these, and referring to Figure 3, we now can write the following equations,

$$\begin{gathered}\|\mathbf{d}\|^2 = D^2 + l^2\sin^2\phi \\ \hat{\mathbf{n}} = \cos\beta\ \hat{\mathbf{k}} - \sin\beta\ \hat{\mathbf{j}}\ , \quad \mathrm{d}\mathbf{r} = (\cos\phi\ \hat{\mathbf{i}} + \sin\phi\ \hat{\mathbf{j}})\ \mathrm{d}l\end{gathered} \tag{9}$$

where, ϕ is a constant angle between S_i and T on the plane of the paper, $\cos\beta = \frac{l\sin\phi}{\|\mathbf{d}\|}$, $\sin\beta = \frac{D}{\|\mathbf{d}\|}$, and l is the length parameter along T starting at $\mathbf{q}$. Thus from (8),

$$\mathbf{B}_i\cdot \mathrm{d}\mathbf{r} = -\frac{1}{2\pi}\frac{\sin\beta\ \sin\phi}{\|\mathbf{d}\|}\mathrm{d}l = \ -\frac{D\sin\phi}{2\pi}\frac{\mathrm{d}l}{D^2 + l^2\sin^2\phi} \tag{10}$$

Thus,

$$\begin{aligned}\int_{\overline{\tau}}\mathbf{B}_i\cdot \mathrm{d}\mathbf{r} &= -\frac{D\sin\phi}{2\pi}\int_{l_A}^{l_B}\frac{\mathrm{d}l}{D^2+l^2\sin^2\phi} \\ &= -\frac{1}{2\pi}\left(\arctan\left(\frac{l_B}{D/\sin\phi}\right) - \arctan\left(\frac{l_A}{D/\sin\phi}\right)\right)\end{aligned} \tag{11}$$

An arctangent of a quantity, with consideration for proper quadrants, can assume values between $-\pi$ and π. Thus the quantity within the outer brackets of Equation (11), that is the difference of two arctangents, can assume values between -2π and 2π. Thus the integral $\int_{\overline{\tau}}\mathbf{B}_i\cdot \mathrm{d}\mathbf{r}$, can assume values between -1 and 1. Thus, as claimed in Section -B, a straight line segment trajectory indeed has the value of $h_i(\tau)$ in $(-1, 1)$ for this simple case of infinitely long line S_i.

REFERENCES

[1] David J. Griffiths. *Introduction to Electrodynamics (3rd Edition)*. Benjamin Cummings, 1998.